AC Steady-State Power

* When dealing with power generation + distribution, we are often interested in steady-state performance of systems excited by sinusoidal signals

* One important quantity is the power generated or dissipated in a given element.

* The instantaneous power dissipated in this element is

$$p(t) = v(t)\, i(t)$$

in units of Watts

* However, this quantity can vary greatly in time

* If the element is linear, then if the voltage is periodic with period T, ie if $v(t+T) = v(t)$, then the current is also periodic with period T

in that case,

$$p(t+T) = v(t+T)\, i(t+T) = p(t)$$

$\Rightarrow$ power is periodic with period T

* In that case, we are often interested in the average power

$$P_{av} = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\,dt,$$

where t_0 is arbitrary

* If the voltage is sinusoidal,

$$v(t) = V_m \cos(\omega t + \phi)$$

If the ~~circuit~~ element is linear, then in the steady state,

$$i(t) = I_m \cos(\omega t + \theta)$$

* These signals are periodic with period $T = \frac{2\pi}{\omega}$

* The instantaneous power is

$$p(t) = V_m I_m \cos(\omega t + \phi)\cos(\omega t + \theta)$$

$$= \frac{V_m I_m}{2}\left[\cos(\phi - \theta) + \cos(2\omega t + \phi + \theta)\right]$$

* Choosing $t_0 = 0$,

$$P_{av} = \frac{1}{T}\int_0^T p(t)\,dt$$

$$= \frac{V_m I_m \cos(\phi - \theta)}{2T}\int_0^T dt$$

$$+ \frac{V_m I_m}{2T}\int_0^T \cos(2\omega t + \theta + \phi)\,dt$$

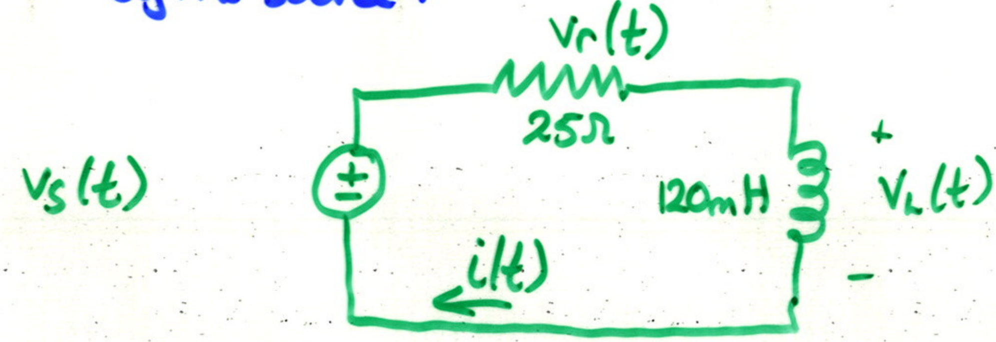
The second integral is the integral of a cosine over two periods, + hence is zero

Therefore

$$P_{av} = \frac{V_m I_m}{2} \cos(\phi - \theta)$$

EXAMPLE

Find the average power delivered to the resistor and inductor. What is the average power generated by the source?



$$v_s(t) = 20 \cos(100t - 15^\circ) \text{ V}$$

Using phasor analysis, + voltage division,

$$v_r(t) = 18 \cos(100t - 41^\circ)$$

$$v_L(t) = 8.66 \cos(100t + 49^\circ)$$

$$i(t) = 721 \times 10^{-3} \cos(100t - 41^\circ)$$

Resistor

$$P_{av,r} = \frac{18 \times 0.721}{2} \cos(-41^\circ - (-41^\circ))$$

$$= 6.5\,W$$

Source,

→ note the passive sign convention means that power will be negative, because the source <u>generates</u> power

$$P_{s,\text{generated}} = \frac{20 \times 0.721}{2} \cos(-15^\circ - (-41^\circ)) = 6.5\,W$$

Inductor –

$$P_{av,L} = \frac{8.66 \times 0.721}{2} \cos(49^\circ - (-41^\circ))$$

$$= 0\,!$$

Why is this zero?

The voltage and current are 90° out of phase! The inductor absorbs/stores power on some part of the cycle, + provides power on other parts. These average out to zero.

Effective Values of periodic waveforms.

Given a periodic ~~so~~ current, $i(t)$, the average power ~~that~~ delivered to a resistor R is

$$P_{av} = \frac{1}{T}\int_0^T i^2(t)\,R\,dt.$$

The effective value of $i(t)$ is defined by

$$I_{eff} = \sqrt{\frac{P_{av}}{R}} \qquad (*)$$

That is, ~~the~~

$$P_{av} = \frac{1}{T}\int_0^T i^2(t)R\,dt = I_{eff}^2 R.$$

Solving $(*)$ we have that

$$I_{eff} = \left(\frac{1}{T}\int_0^T i^2(t)\,dt\right)^{1/2}$$

for this reason I_{eff}, is often called the root mean square (RMS) value, I_{rms}

Similarly, for a periodic voltage signal,

$$V_{rms} = \left(\frac{1}{T}\int_0^T v^2(t)\,dt\right)^{1/2}$$

For a sinusoidal signal $s(t) = A\cos(\omega t + \phi)$

$$S_{rms} = \left(\frac{1}{T}\int_0^T A^2\cos^2(\omega t+\phi)\,dt\right)^{\frac{1}{2}}$$

$$= \frac{A}{\sqrt{2}}$$

Complex power

* It is often convenient to anlyse the steady-state response of a linear circuit to a sinusoidal excitation in the "phasor domain"

* However to calculate power, we need to return to the time domain.

* An alternative approach is to define a notion of power in the phasor domain.

* Let there be a sinusoidal excitation of an element at one frequency, ω.

  Let $\underline{I}(\omega) = I_m e^{j\theta}$ and $\underline{V}(\omega) = V_m e^{j\phi}$ be the current and voltage phasors, defined according to the passive sign convention

  Then, the complex power delivered to the element is

$$S = \frac{\underline{V}(\omega)\, \underline{I}(\omega)^*}{2} = \frac{V_m e^{j\phi} . I_m e^{-j\theta}}{2}.$$

$$= \frac{V_m I_m}{2} e^{j(\phi-\theta)} = \frac{V_m I_m}{2} \angle \phi - \theta$$

The asterisk denotes complex conjugation

The apparent power is

$$|S| = \frac{V_m I_m}{2}$$

* In rectangular coordinates,

$$S = \underbrace{\frac{V_m I_m}{2} \cos(\phi - \theta)} + j \frac{V_m I_m}{2} \sin(\phi - \theta)$$

This real part is the average power

The imaginary part is called the reactive power

$$S = P_{av} + jQ.$$

P_{av} is measured in ~~Volt~~ Watts (W)

S is measured in Volt-Amps (VA)

Q is measured in Volt-Amps-Reactive (VAR)

Given that the impedance of an element is

$$\underline{Z}(\omega) = \frac{\underline{V}(\omega)}{\underline{I}(\omega)} = R + jX$$

The complex power can be written as (prove for homework)

$$S = \frac{I_m^2}{2} \, Re\{Z\} + j \frac{I_m^2}{2} \, Im\{Z\}$$

Hence $P_{av} = \frac{I_m^2}{2} Re\{Z\}$

Therefore, when the element is a resistor,

$$P_{av} = \frac{I_m^2 R}{2}$$

and when it is a capacitor or inductor

$$P_{av} = 0$$

See Table 11.5-1 for a summary of power relations

Recall.

$$I_{eff} = \frac{I_m}{\sqrt{2}}$$

$$\Rightarrow P_{av} = I_{eff}^2 . R.$$

Conservation of complex power

* The sum of the complex power absorbed by each element of a circuit is zero; ie complex power is conserved

$$\sum_{\substack{\text{all}\\ \text{elements}}} \frac{\underline{V}_k \underline{I}_k^*}{2} = 0 \qquad (*)$$

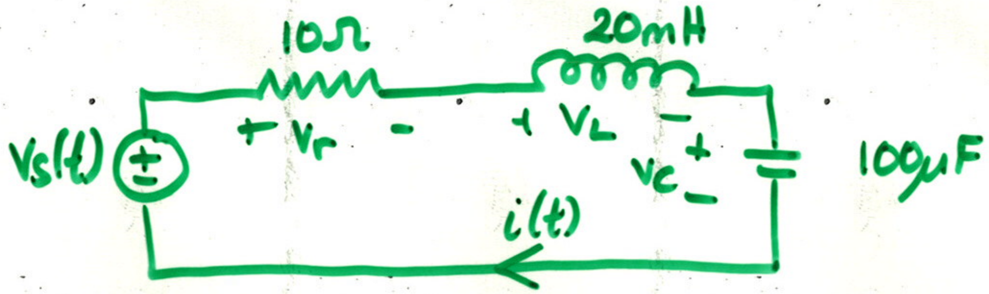
Note that this requires the passive sign convention to be adopted for each element. In that case, the real part of the complex power absorbed by a generating source will be negative

* By taking real + imagniary parts of (*) average power + reactive power are conserved.
ie

$$\sum_{\substack{\text{all}\\ \text{elements}}} P_{av,k} = 0 \qquad \sum_{\substack{\text{all}\\ \text{elements}}} Q_k = 0$$

Example

Show that complex power is conserved in the following circuit when $v_S(t) = 100\cos 1000t$



Using phasor analysis

$$\underline{V_S} = 100$$

$$\underline{I}(\omega) = \frac{\underline{V_S}}{\underline{Z}(\omega)} = \frac{100}{10 + j\omega L - j\frac{1}{\omega C}}$$

$$= 7.07\, e^{-j45°}$$

Hence,

$$\underline{V_r} = R\underline{I} = 70.7\, e^{-j45°}$$

$$\underline{V_L} = j\omega L \underline{I} = 141.4\, e^{j45°}$$

$$\underline{V_C} = -j\frac{1}{\omega C}\underline{I} = 70.7\, e^{-j185°}$$

* For the source, $\underline{V}_s$ and $\underline{I}$ do not obey the passive sign convention, Hence

$$\underline{S}_s = \frac{\underline{V}_s \underline{I}^*}{2} = 353.5 e^{j45^\circ}$$

is the power generated by the source

* For the other elements, we do satisfy the passive sign convention. Hence.

$$\underline{S}_r = 250$$
$$\underline{S}_L = 500 e^{j90^\circ}$$
$$\underline{S}_C = 250\, e^{-j90^\circ}$$

* If power is conserved, then.

$$\underline{S}_s = \underline{S}_r + \underline{S}_L + \underline{S}_C$$

Right hand side $= 250 + j\,250$

$= 353.5\, e^{j45^\circ}$

$\Rightarrow$ Hence power is conserved.