# Complex Numbers

- Recall the characteristic eqn for a second-order diff eqn.

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

- Solutions:

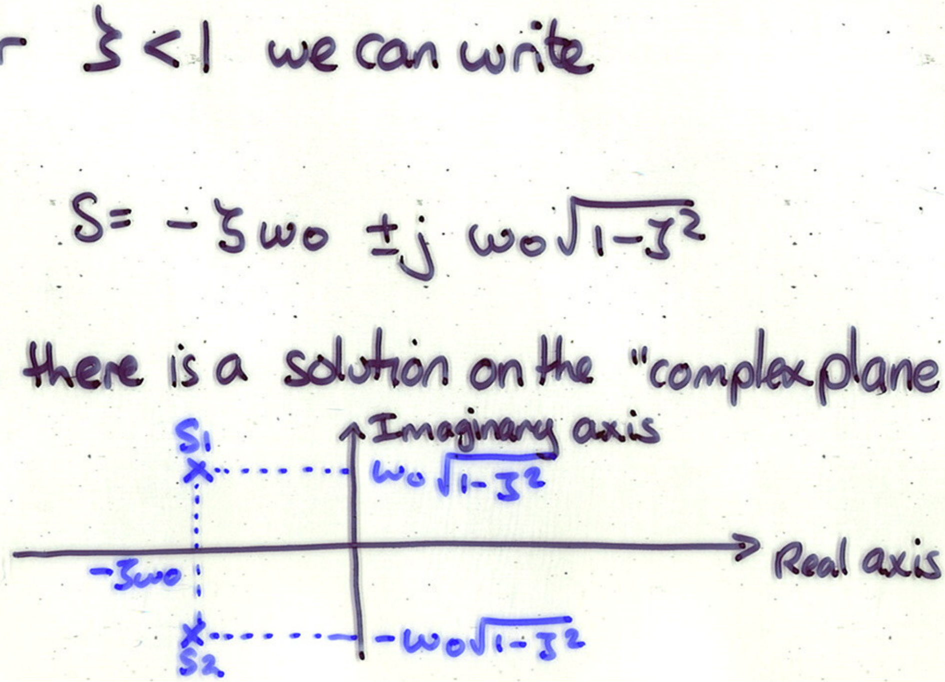
$$s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

- when $\zeta < 1$ no solutions on the real line
- but we have seen that such systems are interesting (defibrillator)
- We would like to have math tools to deal with this.

- If we define $j$ to be the object for which $j^2 = -1$ then for $\zeta < 1$ we can write

$$s = -\zeta\omega_0 \pm j\,\omega_0\sqrt{1-\zeta^2}$$

Now there is a solution on the "complex plane"



- We could represent these points by two-element vectors

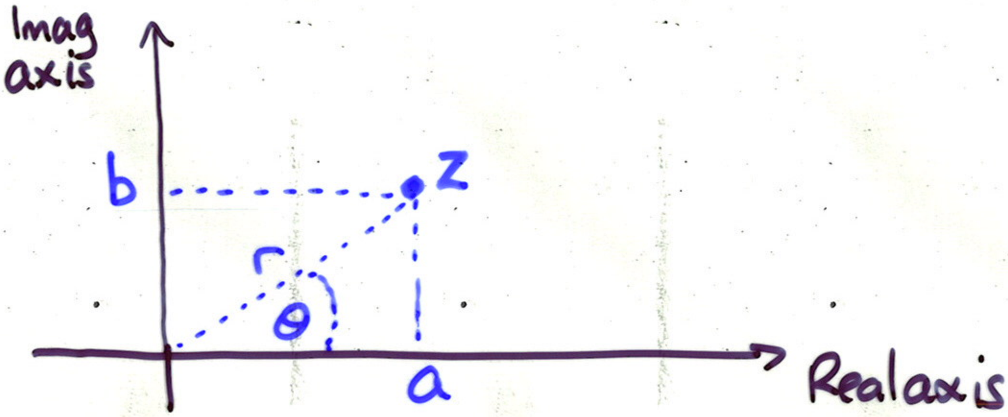
$$\begin{bmatrix} -\zeta\omega_0 \\ \omega_0\sqrt{1-\zeta^2} \end{bmatrix} \text{ and } \begin{bmatrix} -\zeta\omega_0 \\ -\omega_0\sqrt{1-\zeta^2} \end{bmatrix}.$$

and then ~~we~~ use vector and matrix algebra

- clumsy
- impedes insight

- Better to develop a "complex" number system

Consider a complex number $z = a + jb$




$a$ is called the real part, $a = \text{Re}\{z\}$

$b$ is called the imaginary part, $b = \text{Im}\{z\}$

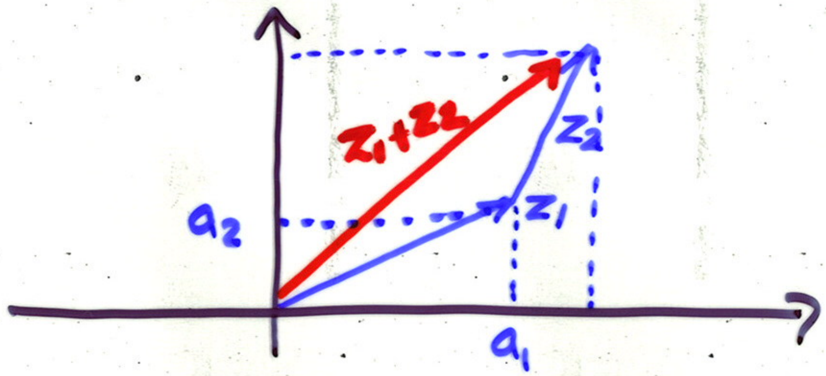
$r = \sqrt{a^2 + b^2}$ is called the magnitude

$\theta = \text{Atan}(b, a)$ is called the phase

when $a > 0$, $\theta = \text{atan}(b/a)$

## Addition

What is $Z_1 + Z_2$ ?




$$Z_s = Z_1 + Z_2$$
$$= (a_1 + a_2) + j\,(b_1 + b_2)$$

## Multiplication

$$Z_m = Z_1 Z_2$$
$$= (a_1 + jb_1)(a_2 + jb_2)$$
$$= (a_1 a_2 - b_1 b_2) + j\,(a_1 b_2 + b_1 a_2)$$

Awkward

There ought to be a better way

What is $e^{j\theta}$?

Taylor Series (Maclaurin Series)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

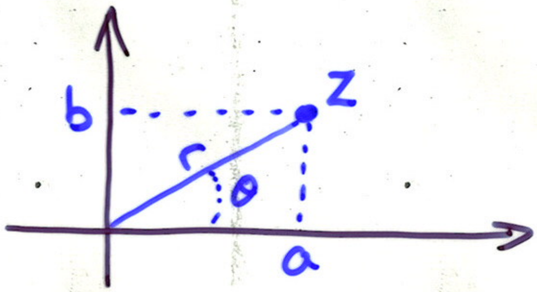
$$\Rightarrow \quad e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right)$$

$$+ j\left(\theta - \frac{\theta^3}{3!} \cdots\right)$$

$$= \cos\theta + j\sin\theta$$

# Multiplication, again




$$z = a + jb = r\cos\theta + j r\sin\theta$$
$$= re^{j\theta}$$

Sometimes written as $r\angle\theta$

Now what is $z_1 z_2$?

$$z_1 z_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2})$$
$$= r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

What about division?

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}$$

$$= \left(\frac{r_1}{r_2}\right) e^{j\theta_1} \cdot e^{-j\theta_2}$$

$$= \left(\frac{r_1}{r_2}\right) e^{j(\theta_1 - \theta_2)}$$

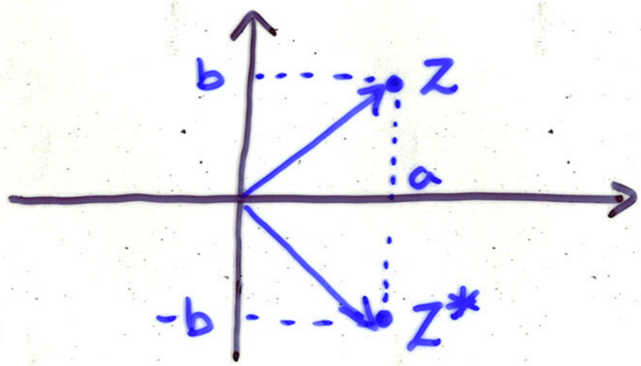
Therefore,

$$z^{-1} = \frac{1}{z} = \left(\frac{1}{r}\right) e^{-j\theta}$$

# Conjugation

$z = a + jb.$

Conjugate: $z^* = a - jb.$




Therefore, if $z = re^{j\theta}$, $z^* = re^{-j\theta}$

Properties

$$\frac{z+z^*}{2} = \frac{a+jb+a-jb}{2} = a = \mathrm{Re}\{z\}$$

$$\frac{z-z^*}{2} = \frac{(a+jb)-(a-jb)}{2} = b = \mathrm{Im}\{z\}$$

$$zz^* = re^{j\theta} \cdot re^{-j\theta} = r^2$$

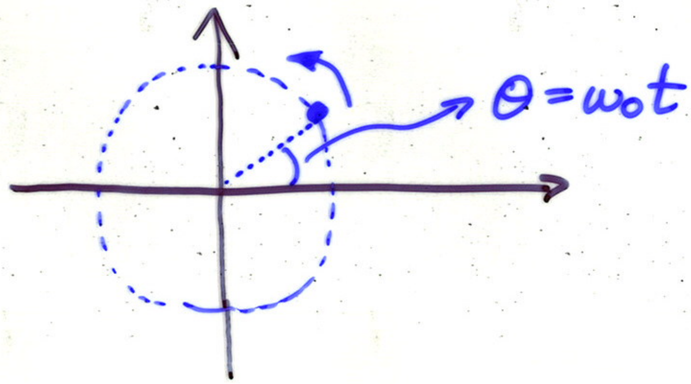
$$\frac{z}{z^*} = \frac{re^{j\theta}}{re^{-j\theta}} = \left(\frac{r}{r}\right) e^{j\theta} \cdot e^{j\theta} = e^{j2\theta}$$

The time function $e^{j\omega_0 t}$

Magnitude = 1

At $t=0$, $z = 1+j0$

Moves around unit circle (anticlockwise)




How long does it take to do a lap?

$\omega_0 t = 2\pi \Rightarrow t = 2\pi/\omega_0$

What about $re^{j(\omega_0 t+\theta)}$ ?

Similar idea

- circle has radius $r$
- at time zero we have $z = e^{j\theta} = \cos\theta + j\sin\theta$

$$re^{j(\omega_0 t+\theta)} = r\cos(\omega_0 t+\theta) + j\, r\sin(\omega_0 t+\theta)$$

Therefore,

$$r\cos(\omega_0 t+\theta) = Re\left\{ re^{j(\omega_0 t+\theta)} \right\}$$

$$= Re\left\{ \underbrace{re^{j\theta}}_{\text{phasor}} \cdot e^{j\omega_0 t} \right\}.$$

## So, how far have we got?

Plot $\frac{1}{1+j\omega}$ against $\omega$

- First write in magnitude & phase form

$$\frac{1}{1+j\omega} = \frac{1}{\sqrt{1+\omega^2}\,.\,e^{j\,\mathrm{atan}(\omega)}}$$

$$= \frac{1}{\sqrt{1+\omega^2}}\; e^{-j\,\mathrm{atan}(\omega)}$$

- **Magnitude**

for small $\omega$, $\frac{1}{\sqrt{1+\omega^2}} \approx 1$

for large $\omega$ $\frac{1}{\sqrt{1+\omega^2}} \approx \frac{1}{\sqrt{\omega^2}} = \frac{1}{\omega}$

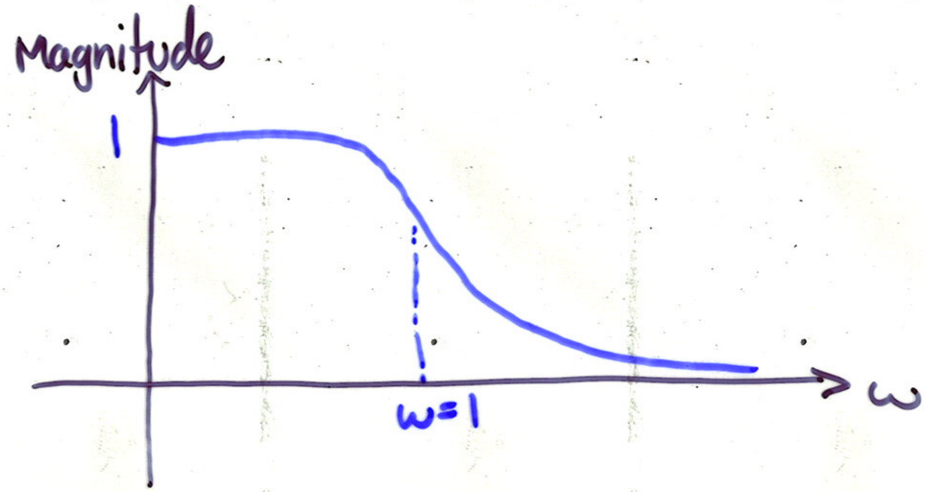
for $\omega = 1$, $\frac{1}{\sqrt{1+\omega^2}} = \frac{1}{\sqrt{2}}$

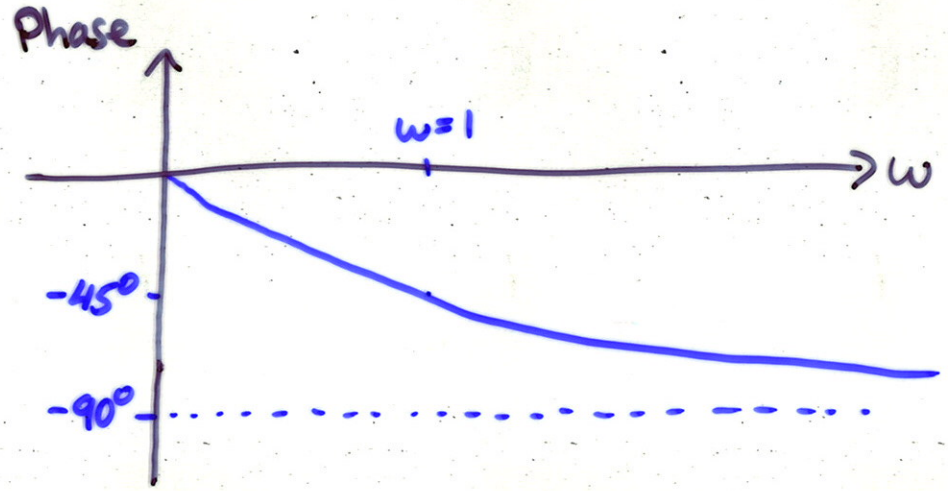
- **Phase**

$\mathrm{atan}(0) = 0$

for large $\omega$, $\mathrm{atan}(\omega) \rightarrow 90^\circ$

$\mathrm{atan}(1) = 45^\circ$


Magnitude
1
ω=1
ω


Phase
ω=1
ω
-45°
-90°