PHASOR ANALYSIS

Phasor representation:

$$A\cos(\omega_0 t+\theta) = \mathrm{Re}\left\{Ae^{j(\omega_0 t+\theta)}\right\} = \mathrm{Re}\left\{Ae^{j\theta}\cdot e^{j\omega_0 t}\right\}$$

$Ae^{j\theta}$ is the phasor representation of $A\cos(\omega_0 t+\theta)$

we must keep ω_0 around as "side information"

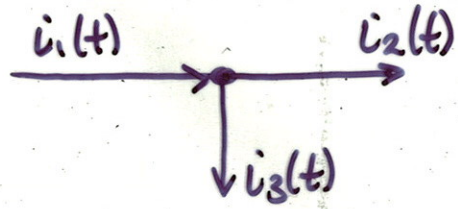
Now: show how phasor representation can be used to simplify circuit analysis when:

1. Circuit has sinusoidal sources, all of the same frequency
2. We are only interested in the steady-state response.

Can handle sinusoidal sources of different frequencies by superposition (in the time domain).

Phasor Relationships

KCL:



At all points in time: $\quad i_1(t) = i_2(t) + i_3(t)$

Therefore, if these are sinusoidal signals

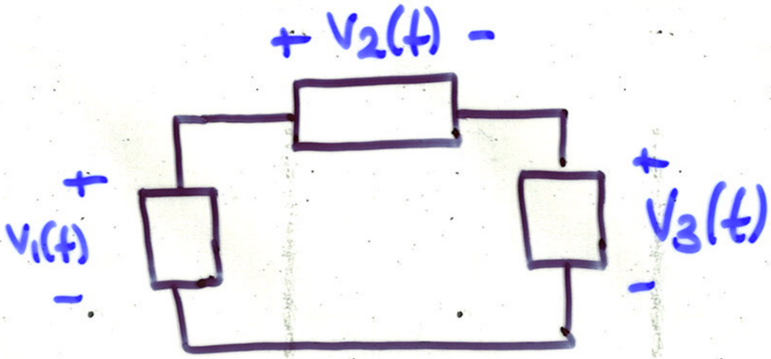
$$A\cos(\omega_0 t + \theta) = B\cos(\omega_0 t + \phi) + C\cos(\omega_0 t + \psi)$$

$$\Rightarrow \quad \mathrm{Re}\{Ae^{j\theta} \cdot e^{j\omega_0 t}\} = \mathrm{Re}\{Be^{j\phi} \cdot e^{j\omega_0 t}\} + \mathrm{Re}\{Ce^{j\psi} \cdot e^{j\omega_0 t}\}$$

$$\Rightarrow \quad Ae^{j\theta} = Be^{j\phi} + Ce^{j\psi}$$

i.e., $\quad \mathbf{I}_1 = \mathbf{I}_2 + \mathbf{I}_3$

KVL:



At all times $\quad -v_1(t) + v_2(t) + v_3(t) = 0$

For sinusoidal signals

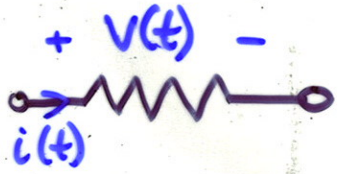
$$-A\cos(\omega_0 t+\theta) + B\cos(\omega_0 t+\phi) + C\cos(\omega_0 t+\psi) = 0$$

$$\Rightarrow -\mathrm{Re}\{Ae^{j\theta}\cdot e^{j\omega_0 t}\} + \mathrm{Re}\{Be^{j\phi}\cdot e^{j\omega_0 t}\} + \mathrm{Re}\{Ce^{j\psi}\cdot e^{j\omega_0 t}\} = 0$$

$$\Rightarrow \quad -Ae^{j\theta} + Be^{j\phi} + Ce^{j\psi} = 0$$

i.e., $\quad -\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = 0$

RESISTORS



$$v(t) = R\,i(t)$$

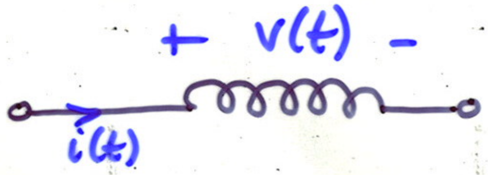
if $i(t) = A\cos(\omega_0 t + \theta)$

$v(t) = RA\cos(\omega_0 t + \theta)$

That is, in phasor domain

$$\mathbf{V} = RAe^{j\theta}$$
$$= R\mathbf{I}$$

INDUCTORS



$$v(t) = L\,\frac{di(t)}{dt}$$

if $\quad i(t) = A\cos(\omega_0 t + \theta)$

$$v(t) = -\omega_0 L\,A\sin(\omega_0 t + \theta)$$

$$= \omega_0 L A\cos(\omega_0 t + \theta + \pi/2)$$

Voltage leads current by 90°

Phasor domain:

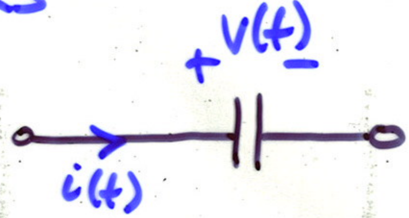
$$\mathbf{V} = \omega_0 L A\,e^{j(\theta + \pi/2)}$$

$$= e^{j\pi/2}\,\omega_0 L A\,e^{j\theta}$$

$$= e^{j\pi/2}\,\omega_0 L\,\mathbf{I}$$

$$= j\omega_0 L\,\mathbf{I}$$

CAPACITORS



$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(x)\,dx$$

if $\quad i(t) = A\cos(\omega_0 t + \theta)$

$$v(t) = \frac{1}{\omega_0 C} A\sin(\omega_0 t + \theta)$$

$$= \frac{1}{\omega_0 C} A\cos(\omega_0 t + \theta - \pi/2)$$

voltage lags current by 90°

Phasor domain:

$$\mathbf{V} = \frac{1}{\omega_0 C} A e^{j(\theta - \pi/2)}$$

$$= \frac{e^{-j\pi/2}}{\omega_0 C} A e^{j\theta}$$

$$= \frac{1}{j\omega_0 C}\mathbf{I}$$

IMPEDANCE

For resistors, inductors and capacitors, we have seen

$$\mathbf{V} = \mathbf{Z}\,\mathbf{I}$$

Like Ohm's Law, but in the phasor domain

$\mathbf{Z}$ called impedance

In general, $Z = R + jX$

R: Resistance
X: Reactance

We also define admittance

$$\mathbf{Y} = \mathbf{I}/\mathbf{V}$$

$$= G + jB$$

G: conductance
B: susceptance

$$Y = \frac{1}{Z} = \frac{1}{R+jX}$$

$$= \frac{1}{R+jX} \cdot \frac{R-jX}{R-jX} = \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2}$$

Note that $G \neq \frac{1}{R}$, $B \neq 1/X$

except in special cases

What have we achieved?

- KCL, KVL work in the phasor domain
- For resistors, inductors and capacitors,

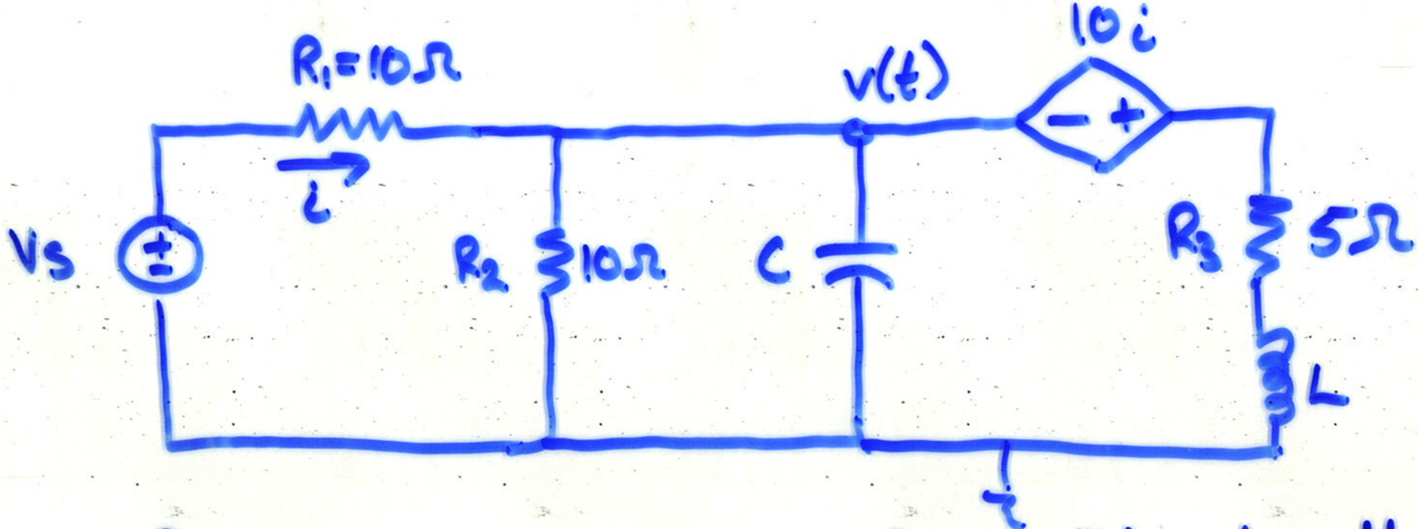
  $$\dot{V} = Z I$$

- Therefore
  - Node, Mesh, Thevenin, Norton, Superposition apply directly
  - Impedances in series add.
  - Admittances in parallel add

- As a result:
  - for sinusoidal sources of one frequency, finding the steady-state solution is only as hard as circuits with DC sources and resistors

    but algebra is complex rather than real

* Here we'll just do an example:



If $L = 0.5\ H$, $C = 10\,mF$, find $v(t)$ when the circuit has reached the steady-state, if

$v_s(t) = 10 \cos(10t)$ Volts.

Step 1 - Identify supernode for dependent source

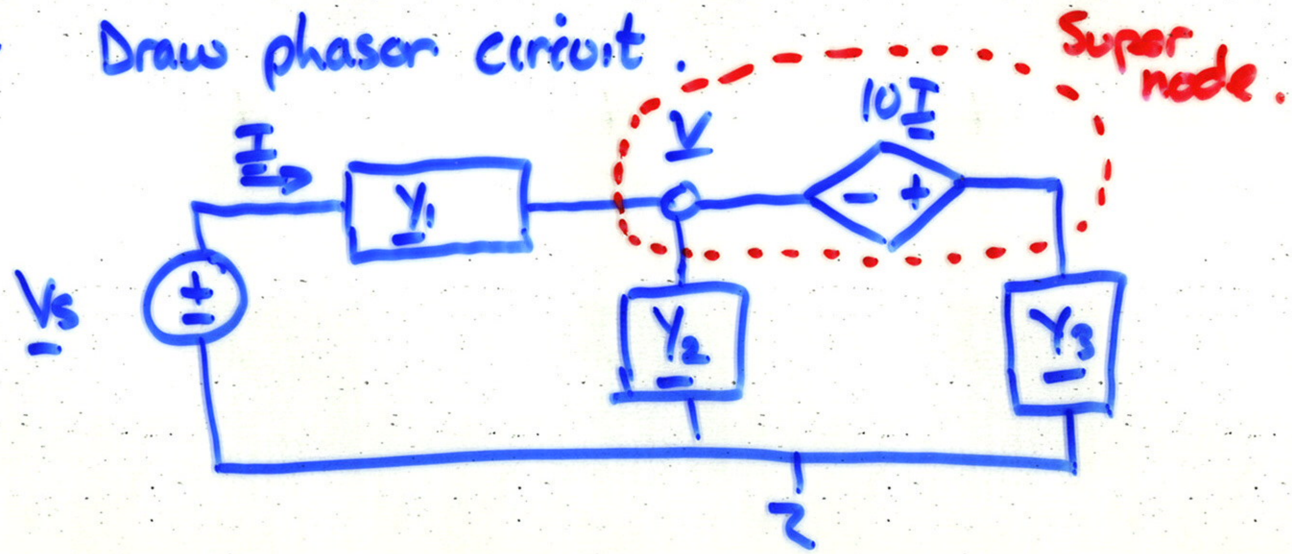
Step 2 - observe that $i(t) = \dfrac{v_S(t) - v(t)}{10}$

Step 3 - Calculate impedances

$$Z_L = j\omega L = j5$$

$$Z_C = \frac{1}{j\omega C} = -j10$$

Step 4 - Draw phasor circuit



where $\underline{Y}_1 = \dfrac{1}{R_1} = 0.1$

$$\underline{Y}_2 = \frac{1}{R_2} + \frac{1}{Z_C} = 0.1(1+j)$$

$$\underline{Y}_3 = \frac{1}{R_3 + Z_L} = 0.1(1-j)$$

KCL at supernode

$$Y_1(V-V_S) + Y_2 V + Y_3(V+10I) = 0$$

Dependent source $\quad I = Y_1(V_S - V)$

Re-arrange

$$V = \frac{(Y_1 - 10Y_1Y_3)}{Y_1+Y_2+Y_3-10Y_1Y_3} \cdot V_S$$

Since $V_S = 10e^{j0}$

$$\Rightarrow V = \frac{10j}{2+j}$$

$$= \frac{10}{\sqrt{5}} e^{j\,63.4^\circ}$$

$$\Rightarrow v(t) = \frac{10}{\sqrt{5}} \cos(10t + 63.4^\circ)$$