* [redacted], we introduced some convenient plots of the frequency response of a system.

* Freq. resp. is

$$\underline{H}(\omega) = \frac{\underline{V}_{out}}{\underline{V}_{in}}$$

where $\underline{V}_{out}$ and $\underline{V}_{in}$ are the phasors for frequency ω, of the output + input signals (in this case, voltages).

* We argued that the logarithmic gain in decibels against the log of frequency is often useful; ie.

$$20 \log_{10} \left(|H(\omega)| \right) \quad \text{vs} \quad \log_{10} \omega.$$

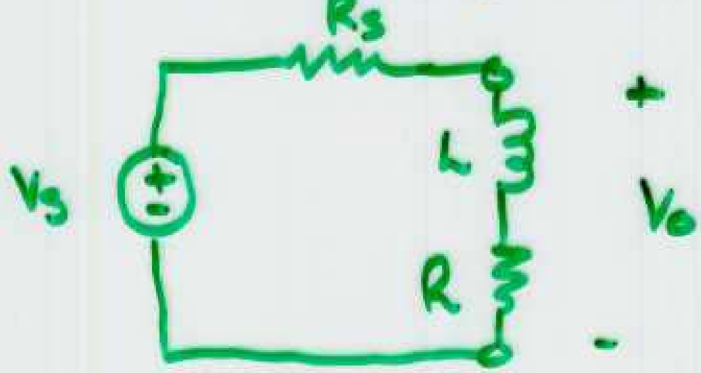
* a complementary phase plot is

$$\phi = \tan^{-1} \left(\frac{\operatorname{Im}\{H(\omega)\}}{\operatorname{Re}\{H(\omega)\}} \right) \quad \text{vs} \quad \log_{10} \omega$$

* We also looked at an example of how the above magnitude plot can be approximated + drawn by hand.

* We will continue the development of that scheme today

lets plot the frequency response of the following circuit.



if v_s is sinusoidal, then using the phasor form of voltage division,

$$\underline{H}(\omega) = \frac{\underline{V_o}}{\underline{V_s}} =$$

$$=$$

where $k = \frac{R}{R+R_s}$, "

$$\omega_1 = \frac{R}{L} < \frac{R+R_s}{L} = \omega_2.$$

k is called "DC gain", note that $\lim_{\omega \to 0} |\underline{H}(\omega)| = k$.

ω_1 is a "corner freq." in the numerator, often called a "zero"

ω_2 is a "corner freq." in the denominator, often called a "pole"

By taking the magnitude,

$$|H(\omega)| = \frac{k\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}}{\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2}}$$

* This is difficult to plot directly, can we approximate it?
* For $\omega \ll \omega_1$, $\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2} \approx 1$

  For $\omega \gg \omega_1$, $\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2} \approx \frac{\omega}{\omega_1}$

  For $\omega \ll \omega_2$, $\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2} \approx 1$

  For $\omega \gg \omega_2$, $\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2} \approx \frac{\omega}{\omega_2}$

By splicing these approximations together, we have

$$|H(\omega)| \approx \begin{cases} k & \omega < \omega_1 \\ k\frac{\omega}{\omega_1} & \omega_1 < \omega < \omega_2 \\ k\frac{\omega_2}{\omega_1} & \omega > \omega_2 \end{cases}$$

What does this do to the logarithmic gain?

$$20\log_{10}(|H(\omega)|) \approx \begin{cases} 20\log_{10} k & \omega < \omega_1 \\ 20(\log_{10} k - \log_{10}(\omega_1)) + 20\log_{10}\omega. & \omega_1 < \omega < \omega_2 \\ 20(\log_{10} k - \log_{10}\omega_1) + 20\log_{10}\omega_2 & \omega > \omega_2 \end{cases}$$

- what do these three pieces look like?
- for $\omega < \omega_1$, and $\omega > \omega_2$, we have a horizontal line
- for $\omega_1 < \omega < \omega_2$, we have an equation of the form.

$$y = c + ax$$

where
$$c = 20(\log_{10} k - \log_{10}\omega_1)$$
$$a = 20$$
$$x = \log_{10}\omega.$$

$\Rightarrow$ a straight line with positive slope.

- Note that k merely shifts the Bode plot up and down it does not change the shape.
  Hence we often plot $20\log_{10}\left(\frac{|H(\omega)|}{k}\right)$

Since $\log_{10}\left(\frac{|H(\omega)|}{k}\right) = \log_{10}(|H(\omega)|) - \log_{10} k$,

$$20\log_{10}\left(\frac{|H(\omega)|}{k}\right) \approx \begin{cases} 0 & \omega < \omega_1 \\ 20\log_{10}\omega - 20\log_{10}\omega_1, & \omega_1 < \omega \le \omega_2 \\ 20\log_{10}\omega_2 - 20\log_{10}\omega_1 & \omega > \omega_2 \end{cases}$$

$$x = 20\log_{10}\left(\frac{\omega_2}{\omega_1}\right)$$

$$H(\omega) = k\,\frac{1 + j\frac{\omega}{\omega_1}}{1 + j\frac{\omega}{\omega_2}}.$$

Now plot the phase of $\underline{H}(\omega)$

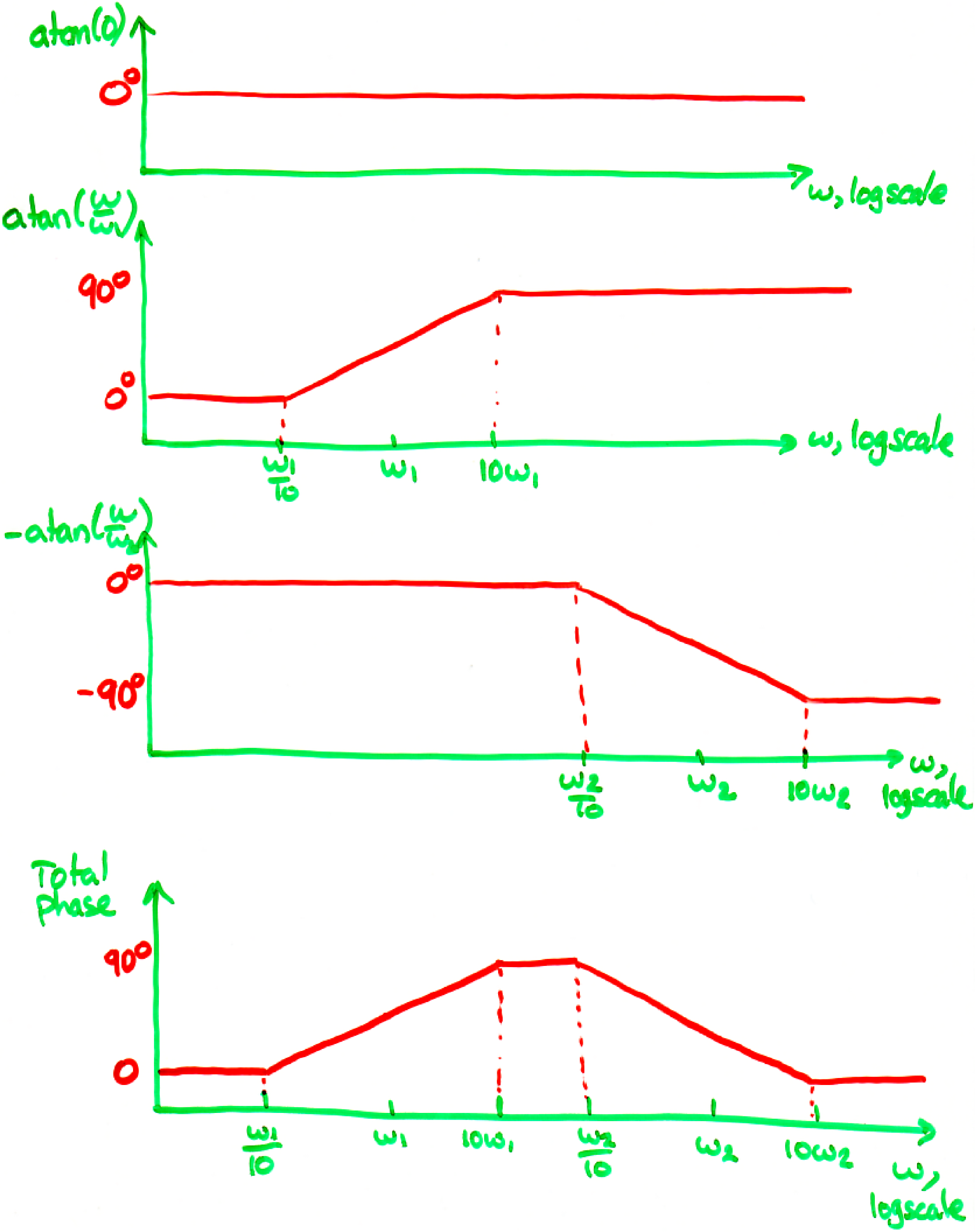
Recall that phase tells us the amount by which sinusoids are advanced or delayed with respect to eachother.

$$\underline{H}(\omega) = k \quad \frac{1+j\,\omega/\omega_1}{1+j\,\omega/\omega_2}$$

$$\angle \underline{H}(\omega) = \operatorname{atan}(0/k) + \operatorname{atan}(\omega/\omega_1) - \operatorname{atan}(\omega/\omega_2)$$

To make the sketch easier, we will assume that $\omega_2 > 100\,\omega_1$

You should try other cases yourselves.



atan(0)
0°
ω, logscale
atan(ω/ω1)
90°
0°
ω, logscale
ω1/10
ω1
10ω1
-atan(ω/ω2)
0°
-90°
ω, logscale
ω2/10
ω2
10ω2
Total phase
90°
0
ω1/10
ω1
10ω1
ω2/10
ω2
10ω2
ω, logscale

The examples we have seen are special cases of general rules.

1. First-order corner frequency in numerator (zero)
   - increases the slope of the Bode magnitude plot by 20dB per decade
   - increases the phase by $90°$

2. First order corner frequency in denominator (pole)
   - decreases the slope of the Bode magnitude plot by 20dB per decade
   - decreases the phase by $90°$

These rules will be formalized soon.

They are the keys to obtaining sketches of the Bode diagrams

Find the asymptotic Bode plot of the following frequency response of a second-order system.

$$H(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j\,2\zeta\omega_0\omega}$$

Approximations:

For $\omega \ll \omega_0$, denominator $\approx \boxed{}$

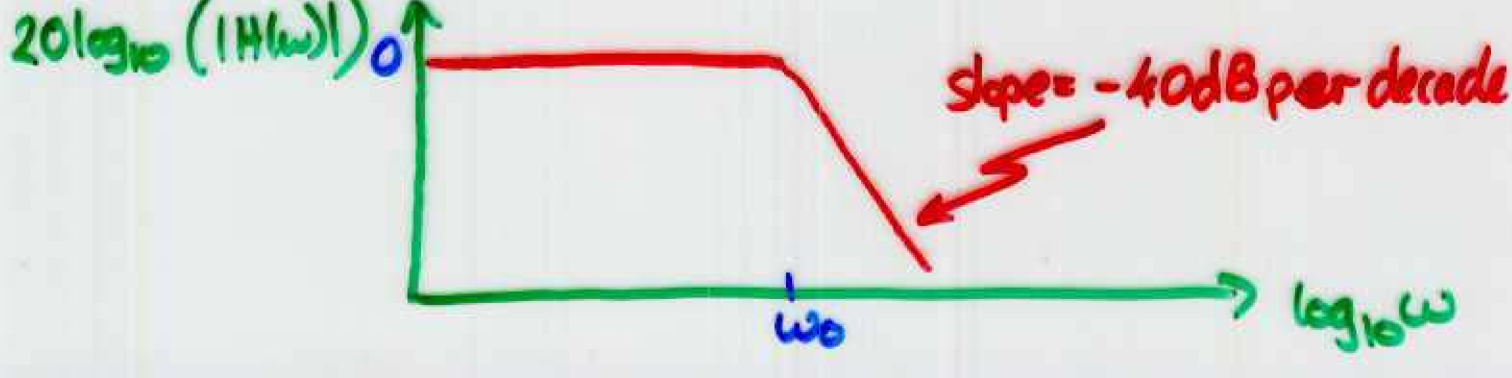
For $\omega \gg \omega_0$, denominator $\approx \boxed{}$

$$\Rightarrow \quad H(\omega) \approx \begin{cases} 1 & \omega < \omega_0 \\ -\dfrac{\omega_0^2}{\omega^2} & \omega > \omega_0 \end{cases}$$

$\Rightarrow$ logarithmic gain

$$20\log_{10}(|H(\omega)|) \approx \begin{cases} 0 & \omega < \omega_0 \\ 40\log_{10}\omega_0 - 40\log_{10}\omega & \omega > \omega_0 \end{cases}$$

Approximate plot.



Now for the phases.

$$\angle \underline{H}(\omega) = -\operatorname{atan}\left(\frac{2\zeta\omega_o\omega}{\omega_o^2-\omega^2}\right)$$

$$= -\operatorname{atan}\left(\frac{2\zeta\,\omega/\omega_o}{1-\omega^2/\omega_o^2}\right)$$

for $\omega \ll \omega_o$

$$\angle \underline{H}(\omega) \approx -\operatorname{atan}(0) = 0^\circ$$

for $\omega \gg \omega_o$

$$\angle \underline{H}(\omega) \approx -\operatorname{atan}\left(\frac{2\zeta\,\omega/\omega_o}{-\omega^2/\omega_o^2}\right)$$

$$= \mp 180^\circ - \operatorname{atan}\left(\frac{2\zeta\,\omega/\omega_o}{-\omega^2/\omega_o^2}\right)$$

$$= \mp 180^\circ - 0^\circ$$

$$= \mp 180^\circ$$

for $\omega = \omega_o$

$$\angle \underline{H}(\omega) = -\operatorname{atan}(\infty)$$

$$= -90^\circ$$

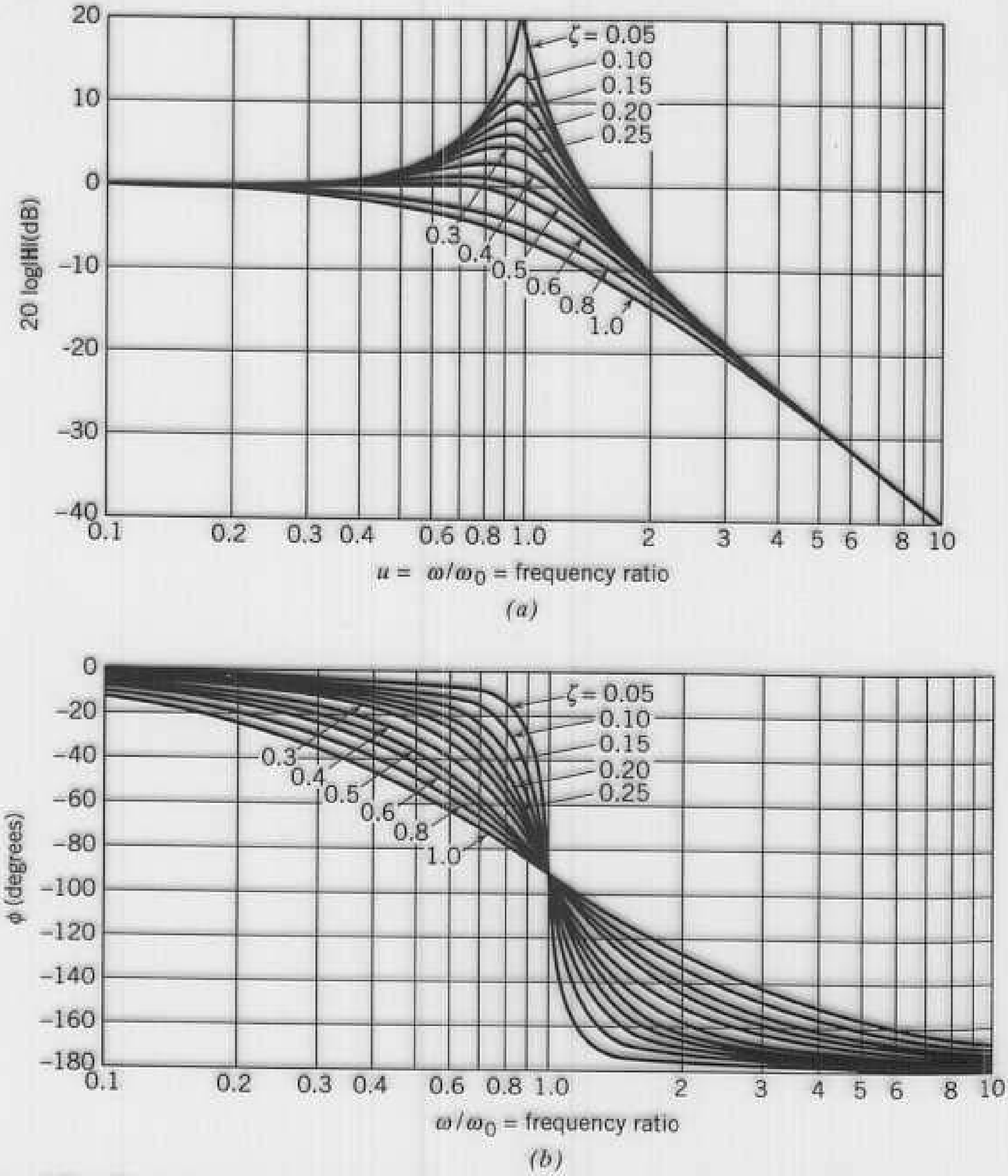


Figure 13.4-7 Bode diagram of $\mathbf{H}(j\omega) = [1 + (2\zeta/\omega_0)j\omega + (j\omega/\omega_0)^2]^{-1}$ for two decades of frequency.

More complicated example

Find asymptotic magnitude Bode plot of

$$H(\omega) = \frac{5(1+0.1j\omega)}{j\omega(1+0.5j\omega)\left(1+0.6\left(\frac{j\omega}{50}\right)-\left(\frac{\omega}{50}\right)^2\right)}$$

Corner frequencies

zeros

poles

Look at low frequencies, $\omega << 2$.

$\Rightarrow \quad H(\omega) \approx \frac{5}{j\omega}$

$\Rightarrow \quad 20\log_{10}(|H(\omega)|) \approx 20\log_{10}5 - 20\log_{10}\omega$

$\Rightarrow$ slope of -20dB per decade.

at $\omega = 1$ it goes through the point 14dB

* at the corner frequency $\omega = 2$ the slope will decrease by 20dB/decade $\Rightarrow$ slope $= -40$dB/decade.

* at corner freq. $\omega = 10$, slope will increase by 20dB/decade $\Rightarrow$ slope $= -20$dB/decade.

* at corner frequency $\omega = 50$, slope will decrease by 40dB per decade, due to ω^2 term

$\Rightarrow$ slope = -60dB per decade.