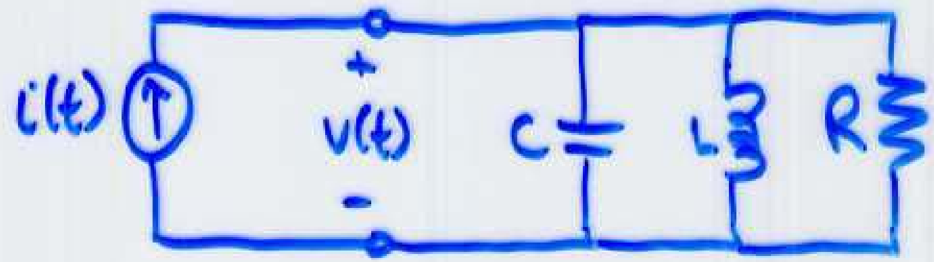
Resonant Circuits.

Consider the following circuit



where

$$i(t) = A\cos\omega t.$$

Since the circuit is linear

$$v(t) = B\cos(\omega t + \theta)$$

and the input impedance is $Z = \frac{B}{A} e^{j\theta}$

In this case,

$Z =$ (impedances in parallel)

$$= \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}} \, e^{-j \tan^{-1} R\left(\omega C - \frac{1}{\omega L}\right)}.$$

what do we observe?

① First of all,

Reactance = 0 when

② Furthermore, magnitude of impedance is a maximum at this point

③ The ω for which $\omega C - \frac{1}{\omega L} = 0$

is $\omega_0 = \frac{1}{\sqrt{LC}}$

④ For $\omega < \omega_0$, phase of Z is positive

$\Rightarrow$ reactance is inductive

For $\omega > \omega_0$, phase of z is negative

$\Rightarrow$ reactance is capacitive

The formula for Z is accurate but inconvenient. We can rewrite it as.

$$Z = \frac{k}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

where

These three parameters characterise the circuit in terms of.

ω_0 = resonant frequency = $\begin{cases} \text{freq. where} \\ \text{reactance} = 0 \end{cases}$

k = maximum impedance

Q = quality factor, measures the rate at which $|z(\omega)|$ decreases from the max at $\omega = \omega_0$

larger $Q \Rightarrow$ faster decay

* An important property of resonant circuits is the bandwidth.

* We will define bandwidth to be $\omega_2 - \omega_1$, where ω_1, ω_2 are the two frequencies where

$$|z(\omega)| = \frac{1}{\sqrt{2}} |z(\omega_0)| = \frac{k}{\sqrt{2}} \qquad (*)$$

* Here we assume that $\omega_1 < \omega_2$

* How do we find ω_1, ω_2? Direct solution of $(*)$
Find ω such that

$$\frac{k}{\sqrt{2}} = \frac{k}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}$$

$$\Leftrightarrow \quad 1 = Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2.$$

$$\Leftrightarrow \quad \pm 1 = Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$\Leftrightarrow \quad \omega^2 \pm \frac{\omega_0 \omega}{Q} - \omega_0^2 = 0$$

This equation has 4 solutions but only 2 are > 0.

$$\omega_{1,2} = \mp \frac{\omega_0}{2Q} + \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 + \omega_0^2}$$

Therefore $\quad BW = \omega_2 - \omega_1 = \frac{\omega_0}{Q} = \frac{1}{RC}$.

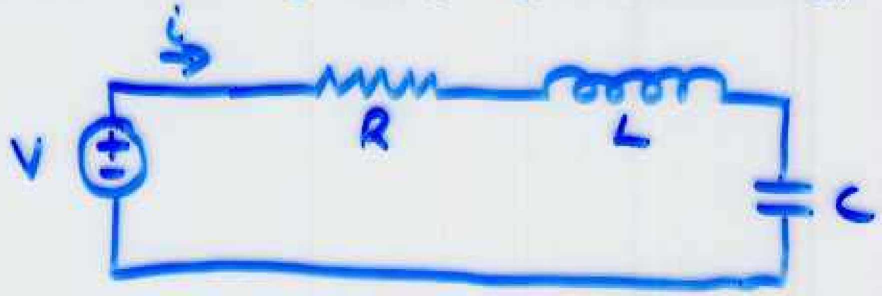
$\Rightarrow$ when Q is larger, then BW is smaller.

Sometimes $1/Q$ is called the relative bandwidth, because

$$\frac{1}{Q} = \frac{BW}{\omega_0} = \frac{\text{bandwidth}}{\text{centre frequency}}$$

We have seen that for a parallel RLC circuit, the impedance is a maximum at the resonant frequency, ω_0, and the rate at which the impedance decays is measured by the quality factor Q.

A dual relationship for the admittance of a series RLC circuit is also true.



Prove for yourself:

$$Y = \frac{k}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$, $k = \frac{1}{R}$

$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$, $BW = \frac{R}{L}$

Now k = max admittance $\Rightarrow$ min impedance occurs at $\omega = \omega_0$

admittance decreases away from ω_0 $\Rightarrow$ impedance incr. away from ω_0.