EE3CL4 C01: Introduction to Linear Control Systems

Section 3: Fundamentals of Feedback

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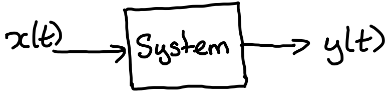
Transfer functions
Closed loop
Stability & Performance
Step response
First-order
Second-order
A taste of pole-placement design
Extensions
Steady-state error
Summary and plan

Outline

Linear Time-Invariant Systems



How do we describe the relationship between $x(t)$ and $y(t)$?

Direct description (time domain):

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_n\frac{d^n x(t)}{dt^n} + b_{n-1}\frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t)$$

- Difficult to solve
- Hard to gain insight

Linear Time-Invariant Systems

Transformed description (Laplace domain), when all init. conds are zero

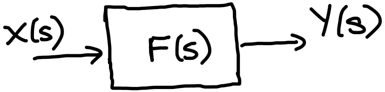
$$s^nY(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_1sY(s) + a_0Y(s)$$
$$= b_ns^nX(s) + b_{n-1}s^{n-1}X(s) + \cdots + b_1sX(s) + b_0X(s)$$

- $Y(s) = F(s)X(s)$, where

$$F(s) = \frac{b_ns^n + b_{n-1}s^{n-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}$$

- Simple to find $Y(s)$; Can then find $y(t)$, if you'd like
- We will do some work so that we can avoid doing that
- We will draw pictures of $y(t)$ and gain insight into $y(t)$ from $F(s)$ and $X(s)$.

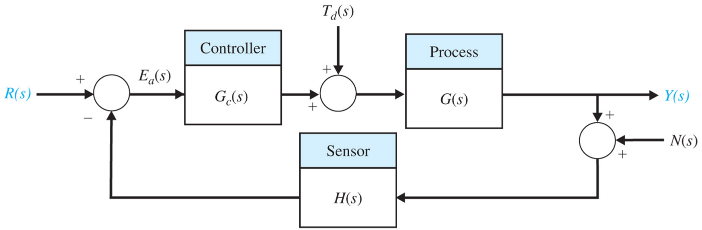
Transfer function



- $Y(s) = F(s)X(s)$
- Stability (more details later):

  the output $y(t)$ is bounded for all bounded inputs $x(t)$
  if and only if
  the poles of $F(s)$ are in the open left half plane

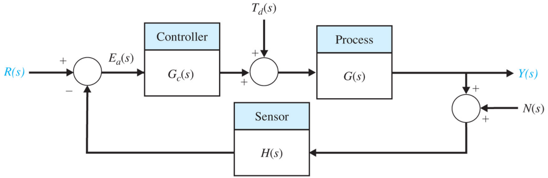
Closed loop control



(b)

- Error: $E(s) = R(s) - Y(s)$
- Measured error: $E_a(s) = R(s) - H(s)\Big(Y(s) + N(s)\Big)$.
- In the general case, $E_a(s) \neq E(s)$.
- When $H(s) = 1$ and $N(s) = 0$, $E_a(s) = E(s)$.

The output signal

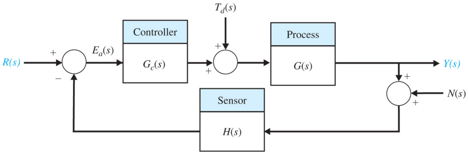


(b)

What is the output $Y(s)$? (Calculate yourself for practice)

$$Y(s) = \frac{G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} R(s) + \frac{G(s)}{1 + H(s)G_c(s)G(s)} T_d(s) - \frac{H(s)G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} N(s)$$

The error signal, $H(s) = 1$



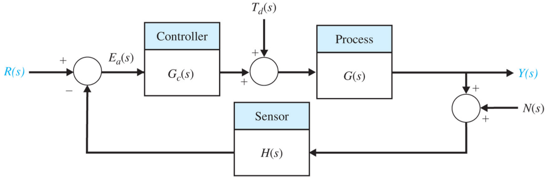
(b)

What is the error $E(s) = R(s) - Y(s)$?
To simplify things, consider the case where $H(s) = 1$

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s) - \frac{G(s)}{1 + G_c(s)G(s)} T_d(s) + \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} N(s)$$

Recall, $E_a(s) = E(s)$ only if $H(s) = 1$ *and* $N(s) = 0$.

Loop gain, $H(s) = 1$



(b)

Define loop gain: $L(s) = G_c(s)G(s)$

$$E(s) = \frac{1}{1 + L(s)} R(s) - \frac{G(s)}{1 + L(s)} T_d(s) + \frac{L(s)}{1 + L(s)} N(s)$$

$G(s)$ is fixed, but we can design $G_c(s)$

What insight can we gain into how to design $G_c(s)$?

Stability, $H(s) = 1$

$$E(s) = \frac{1}{1 + L(s)} R(s) - \frac{G(s)}{1 + L(s)} T_d(s) + \frac{L(s)}{1 + L(s)} N(s)$$

- Stability: bounded inputs lead to bounded errors
  poles of transfer function in left half plane
- For simplicity, let $T_d(s) = 0$, $N(s) = 0$
- $G(s) = \frac{n_G(s)}{d_G(s)}$; $G_c(s) = \frac{n_C(s)}{d_C(s)}$; $L(s) = \frac{n_C(s)}{d_C(s)} \frac{n_G(s)}{d_G(s)}$
- Hence,
$$\frac{1}{1 + L(s)} = \frac{d_C(s) d_G(s)}{d_C(s) d_G(s) + n_C(s) n_G(s)}$$
- $\implies$ closed loop poles are roots of $d_C(s) d_G(s) + n_C(s) n_G(s)$
- These can be in left half plane even if $G(s)$ is unstable, but they can also be in the right half plane if $G(s)$ is stable

Performance: s-domain, $H(s) = 1$

$$E(s) = \frac{1}{1+L(s)}R(s) - \frac{G(s)}{1+L(s)}T_d(s) + \frac{L(s)}{1+L(s)}N(s)$$

What else do we want, in addition to stability?

- Good tracking: $E(s)$ depends only weakly on $R(s)$
  $\implies$ $L(s)$ large where $R(s)$ large
- Good disturbance rejection:
  $\implies$ $L(s)$ large where $T_d(s)$ large
- Good noise suppression:
  $\implies$ $L(s)$ small where $N(s)$ large

A taste of loop shaping, $H(s) = 1$

Possibly easier to understand in pure freq. domain, $s = j\omega$

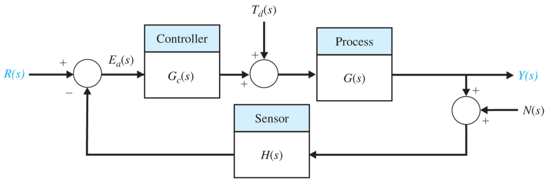
Recall that $L(s) = G_c(s)G(s)$,
$G(s)$: fixed; $G_c(s)$: controller to be designed

- Good tracking: $\implies$ $L(s)$ large where $R(s)$ large
  $|L(j\omega)|$ large in the important frequency bands of $r(t)$
- Good dist. rejection: $\implies$ $L(s)$ large where $T_d(s)$ large
  $|L(j\omega)|$ large in the important frequency bands of $t_d(t)$
- Good noise suppr.: $\implies$ $L(s)$ small where $N(s)$ large
  $|L(j\omega)|$ small in the important frequency bands of $n(t)$

Typically, $L(j\omega)$ is a low-pass function,

Any constraints? Stability! Limits how fast we transition from pass band to stop band of low pass function (more later). Any others?

Inherent constraints, $H(s) = 1$



(b)

Define sensitivity: $S(s) = \dfrac{1}{1 + L(s)}$

Define complementary sensitivity: $C(s) = \dfrac{L(s)}{1 + L(s)}$

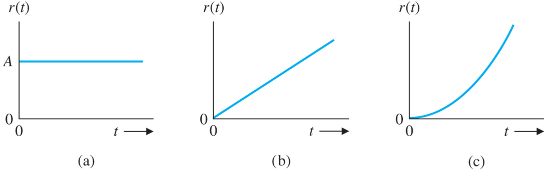
$$E(s) = S(s)R(s) - S(s)G(s)T_d(s) + C(s)N(s)$$

Note that $S(s) + C(s) = 1$.

Trading $S(s)$ against $C(s)$, with stability,
is a key part of the art of control design

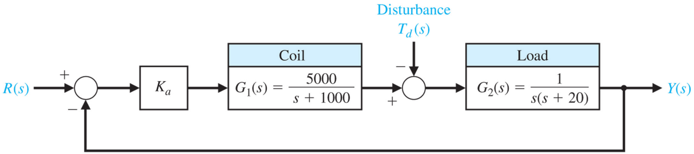
Performance: time-domain

- Trade-offs in time-domain performance are also a key part of the art of control design
- Difficult for arbitrary inputs
- In classical control techniques, typically assessed via
  - nature of transient component of step response
    - how fast does system respond?
    - how long does it take to settle to new operating point
  - steady-state error for constant changes in position, or velocity or acceleration; that is steady-state error for
    - step input; ramp input, parabolic input



Trade-off example

Let's briefly examine some of those design trade-offs using the disk drive system

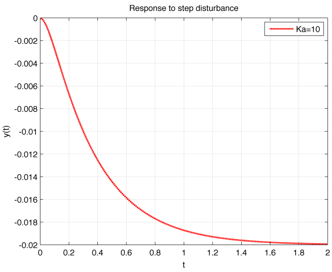


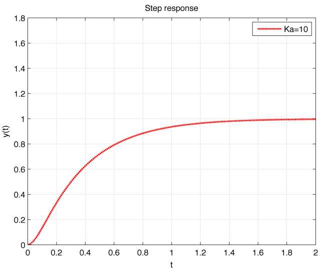
$$Y(s) = \frac{5000K_a}{s^3 + 1020s^2 + 20000s + 5000K_a} R(s) + \frac{s + 1000}{s^3 + 1020s^2 + 20000s + 5000K_a} T_d(s)$$

Coarsely design K_a to balance properties of step response and response to step disturbance

Responses for $K_a = 10$

Disturbance step response and step response



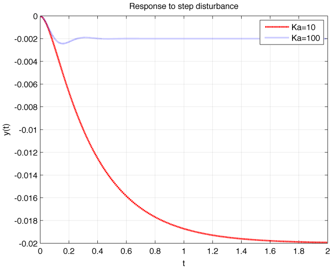


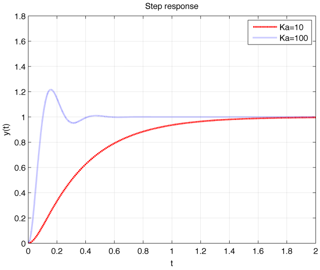
Low gain:

- steady-state disturbance might not be negligible
- slow transient response for step input

Responses for $K_a = 10, 100$

Disturbance step response and step response



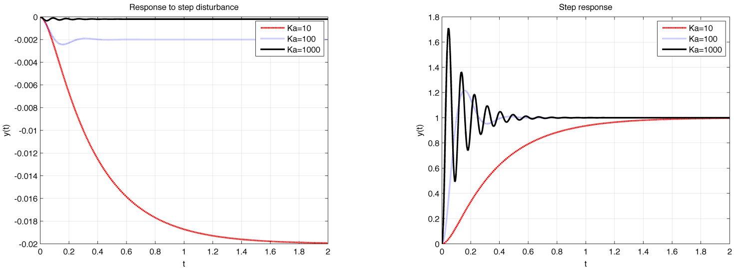


Medium gain:

- steady-state disturbance much reduced
- faster transient response for step input, but now some overshoot

Responses for $K_a = 10, 100, 1000$

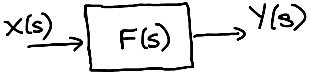
Disturbance step response and step response



High gain:

- steady-state disturbance almost completely rejected
- fast transient response for step input, but now significant overshoot
- Actually can show by Routh Hurwitz technique (later) that loop is unstable for $K_a \geq 4080$

Step response



- As earlier, the step response is the time-domain output of a system that is initially at rest (zero initial conditions), when the input is a unit step function
- We can compute this directly from the differential equation, if we would like to do that
- Alternatively, we can compute it using Laplace transforms:

$$y_{\text{step\_resp}}(t) = \mathcal{L}^{-1}\left(F(s)\frac{1}{s}\right)$$

where $\mathcal{L}^{-1}(\cdot)$ represents the inverse Laplace transform

A first-order system

- Consider the first-order system $F(s) = F_1(s) = \frac{p_1}{s+p_1}$
- For step response,

$$Y_{\text{step\_resp},F_1}(s) = \frac{p_1}{s(s+p_1)} = \frac{1}{s} - \frac{1}{s+p_1}$$

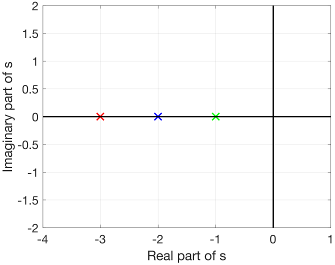
- Hence,

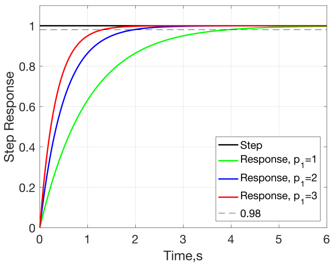
$$y_{\text{step\_resp},F_1}(t) = 1 - e^{-p_1 t}$$

- Note that speed of response depends on pole position

Pole positions and responses

$$Y_{\text{step\_resp},F_1}(s) = \frac{p_1}{s(s+p_1)}$$





Response time

- How long does it take to get there? Forever!
- How long does it take to get close? Say 98%
- How long does it take before $y_{\text{step\_resp}}(t) = 1 - e^{-p_1 t} > 0.98$?
- How long does it take before $e^{-p_1 t} < 0.02$?
- We need $t > \log(50)\frac{1}{p_1}$
- Now $\log(50) \approx 4$, so time taken is ≈ 4 time constants
- That is, $\frac{4}{\text{pole position}}$.
- Don't need inverse Laplace to compute this
- Getting within 5% requires around three time constants; i.e., $\frac{3}{\text{pole position}}$

A second-order system

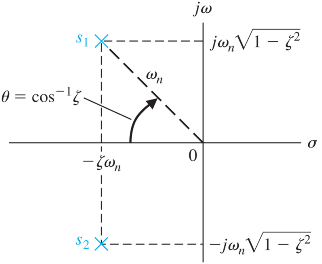
- Second-order system $F(s) = F_2(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
- For step response, $Y_{\text{step\_resp},F_2}(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
- For the case of $\zeta > 1$, system is over-damped
  - System has two real-valued poles, $-p_1, -p_2$.
  - $Y_{\text{step\_resp},F_{2,o}}(s)$ takes the form $\frac{1}{s} - \frac{A}{s+p_1} - \frac{B}{s+p_2}$
  - $y_{\text{step\_resp},F_{2,o}}(t) = 1 - Ae^{-p_1 t} - Be^{-p_2 t}$
  - Pole position insights analogous to first-order case
  - For completeness, $-p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$, $A = \frac{p_2}{p_2-p_1}$, $B = \frac{-p_1}{p_2-p_1}$

A second-order system

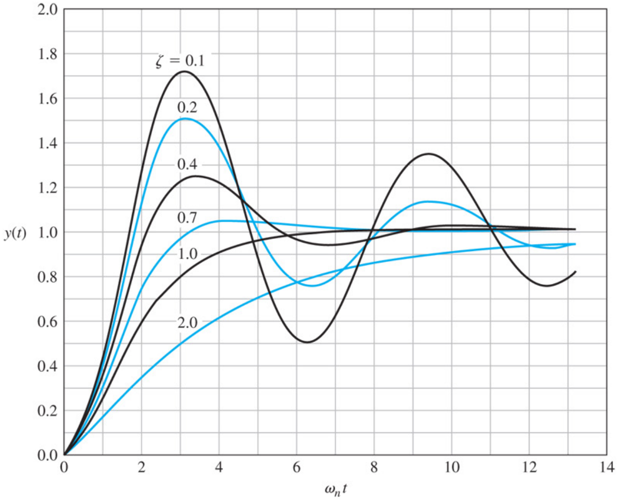
- For the case of $0 < \zeta < 1$, system is under-damped
  - System has a complex-conjugate pair of poles $-p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
  - Step response can be written as

$$y_{\text{step\_resp},F_{2,u}}(t) = 1 - \frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n\beta t + \theta)$$

    where $\beta = \sqrt{1-\zeta^2}$ and $\theta = \cos^{-1}\zeta$.
  - Need new insights; shape depends on pole pos'ns; $s_i = -p_i$

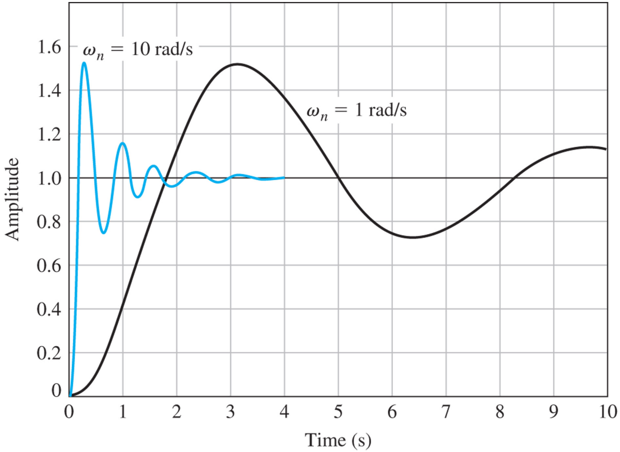


Typical step responses, fixed ω_n



(a)

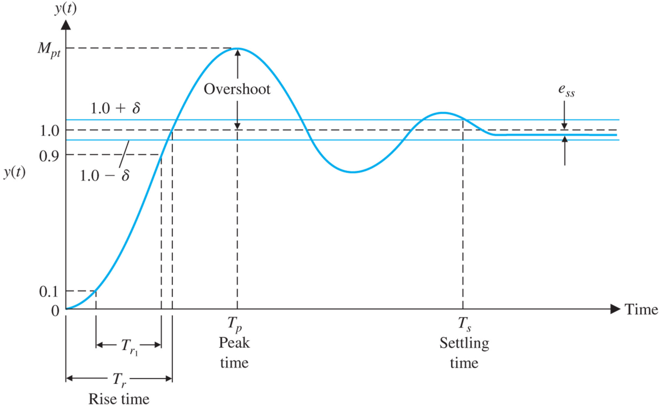
Typical step responses, fixed ζ



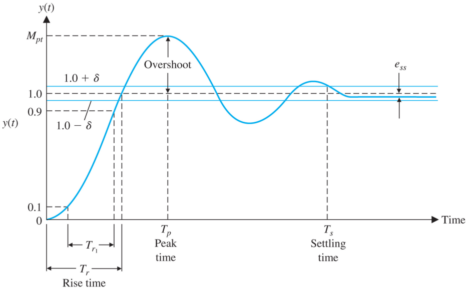
Key parameters of (under-damped) step response

With $\beta = \sqrt{1-\zeta^2}$ and $\theta = \cos^{-1}\zeta$,

$$y_{\text{step\_resp},F_{2,u}}(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t}\sin(\omega_n\beta t + \theta)$$



Peak time and peak value



$$y_{\text{step\_resp},F_{2,u}}(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t}\sin(\omega_n\beta t + \theta)$$

- Peak time: first time $dy(t)/dt = 0$
- Can show that this corresponds to $\omega_n\beta T_p = \pi$
- Hence, $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
- Hence, peak value, $M_{pt} = 1 + e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}$

Percentage overshoot

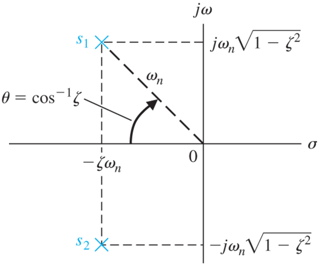
Let fv denote the final value of the step response.

Percentage overshoot defined as: $\text{P.O.} = 100\,\frac{M_{pt} - \text{fv}}{\text{fv}}$

In our example, fv $= 1$, and hence

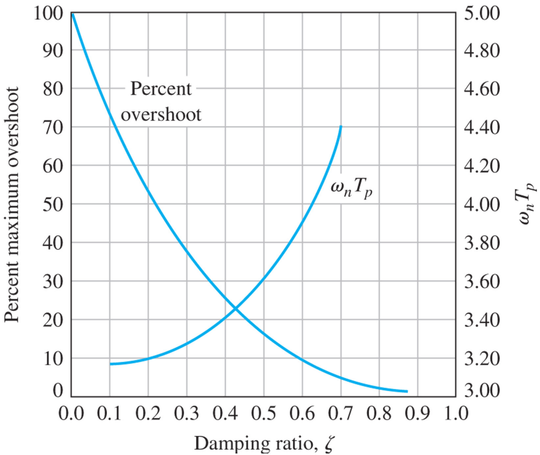
$$\text{P.O.} = 100\, e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}$$

- Depends only on ζ
- That is, depends only on (the cosine of) the angle that the poles make with negative real axis

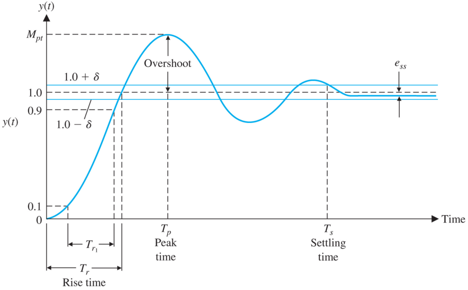


Overshoot vs Peak Time

This is one of the classic trade-offs in control



Steady-state error, e_{ss}, for step input



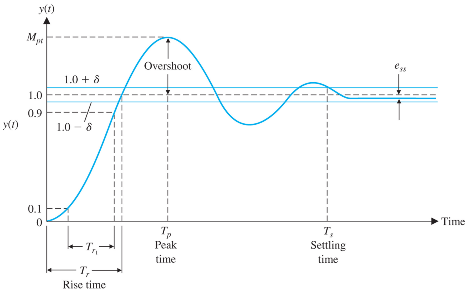
In general this is not zero. (See "Steady-state error" section)

However, for our second-order system,

$$y_{\text{step\_resp},F_{2,u}}(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

Hence $e_{ss} = 0$

Settling time



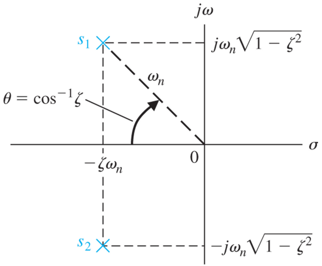
$$y_{\text{step\_resp},,F_{2,u}}(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

- How long does it take to get (and stay) within $\pm x\%$ of final value?
- Tricky.
- Instead, approximate by time constants of envelopes:

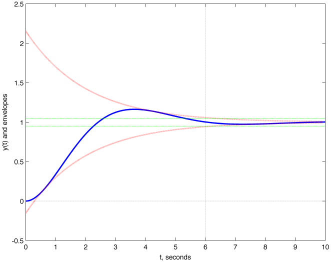
$$1 \pm \frac{1}{\beta} e^{-\zeta\omega_n t}$$

Exponential decay

- We are interested in decay of $e^{-\zeta\omega_n t}$
- We have already seen that in the first-order case
- Decays to around 5% in 3 time constants
  i.e., when $t = \frac{3}{\zeta\omega_n}$, $e^{-\zeta\omega_n t} = 1/e^3 \approx 0.0498 \approx 0.05$
- Decays to around 2% in 4 time constants
  i.e., when $t = \frac{4}{\zeta\omega_n}$, $e^{-\zeta\omega_n t} = 1/e^4 \approx 0.0183 \approx 0.02$
- Time constant is reciprocal of the real part of the poles

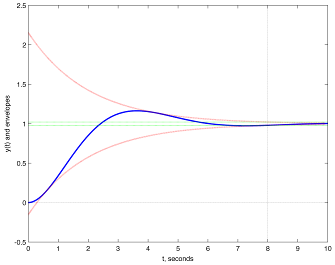


5% settling time



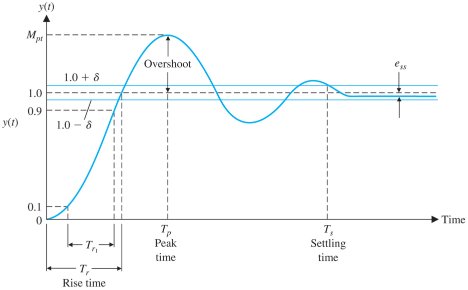
- Green error bounds at ± 0.05.
- $\zeta = 0.5$, $\omega_n = 1$. Hence time constant $= \frac{1}{\zeta\omega_n} = 2$
- After $t = 6$, envelopes are almost within $\pm 5\%$
  Response is within $\pm 5\%$
- $T_{s,5} \approx \frac{3}{\zeta\omega_n}$; approx. good for $\zeta \lesssim 0.9$

2% settling time



- Green error bounds at ± 0.02.
- $\zeta = 0.5$, $\omega_n = 1$. Hence time constant $= \frac{1}{\zeta\omega_n} = 2$
- After $t = 8$, envelopes are almost within $\pm 2\%$
  Response is also almost within $\pm 2\%$
- $T_{s,2} \approx \frac{4}{\zeta\omega_n}$; approx. good for $\zeta \lesssim 0.9$

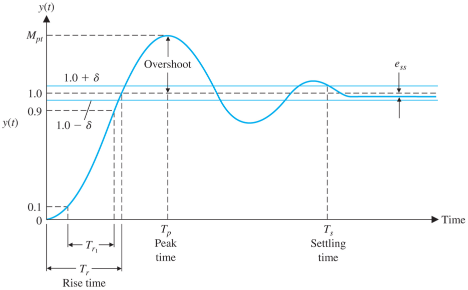
Rise time (under-damped)



$$y_{\text{step\_resp},F_{2,u}}(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

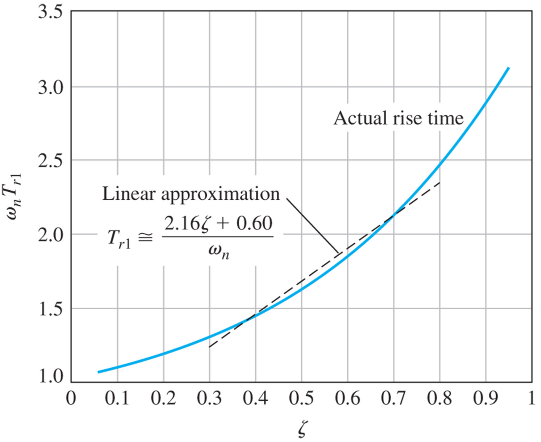
- How long to get to the target (for first time)?
- T_r, the smallest t such that $y(t) = 1$

10%–90% Rise time



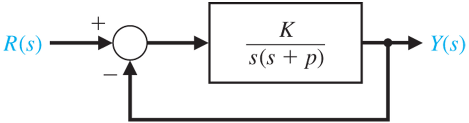
- What is T_r in over-damped case? ∞
- Hence, typically use T_{r1}, the 10%–90% rise time

10%–90% Rise time



- Difficult to get an accurate formula
- Linear approx. for $0.3 \leq \zeta \leq 0.8$ (under-damped),

Design problem



For what values of K and p is the loop under-damped, with

- the 2% settling time ≤ 4 secs, and
- the percentage overshoot $\leq 4.3\%$?

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+ps+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},$$

where $\omega_n = \sqrt{K}$ and $\zeta = p/(2\sqrt{K})$

Pole positions

$$T_{s,2} \approx \frac{4}{\zeta\omega_n} \qquad \text{P.O.} = 100\, e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}$$

- For $T_{s,2} \leq 4$, $\zeta\omega_n \geq 1$
- For P.O. $\leq 4.3\%$, $\zeta \geq 1/\sqrt{2}$

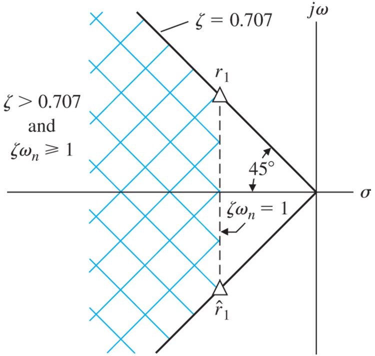
Where should we put the poles of $T(s)$?

Pole positions

$$\zeta\omega_n \geq 1 \qquad \zeta \geq 1/\sqrt{2}$$

$$s_1, \; s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\omega_n\cos(\theta) \pm j\omega_n\sin(\theta)$$

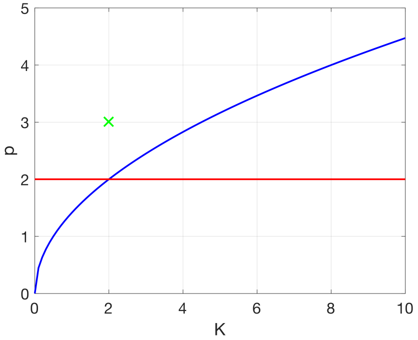
where $\theta = \cos^{-1}(\zeta)$.



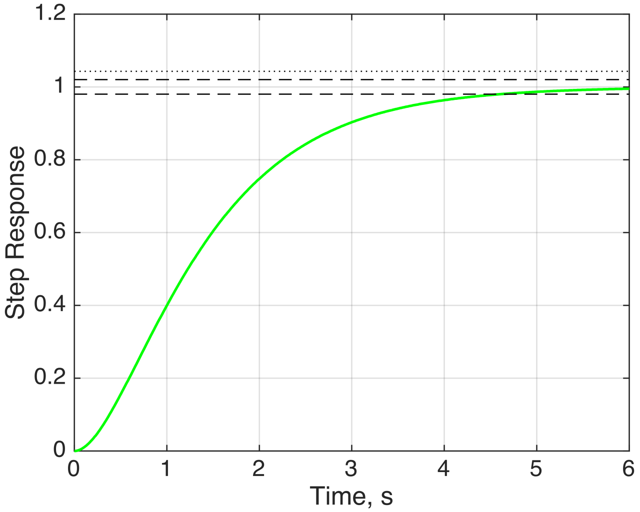
Design constraints

$\zeta\omega_n \geq 1 \qquad \zeta \geq 1/\sqrt{2}$

$p \geq 2 \qquad p \geq \sqrt{2K}$



Design example

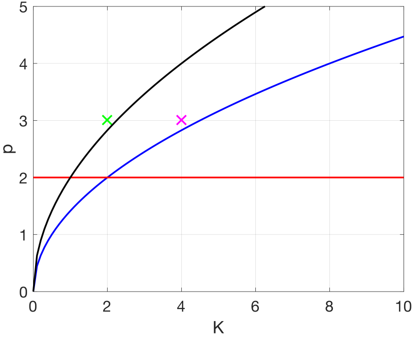


What went wrong?

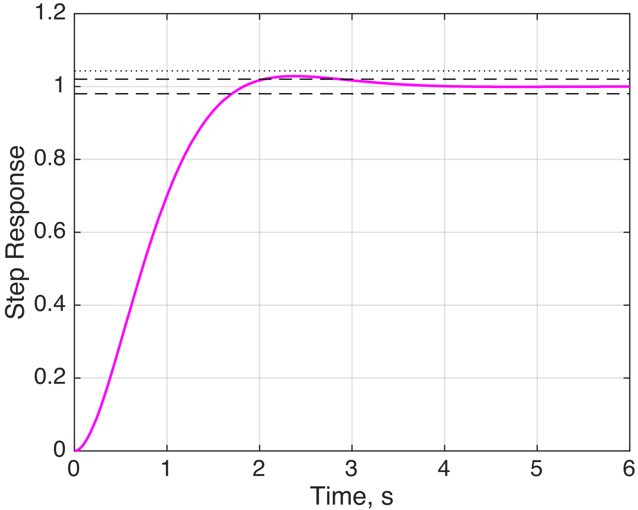
Final design constraints

$$\zeta\omega_n \geq 1 \qquad \zeta \geq 1/\sqrt{2} \qquad \zeta < 1$$

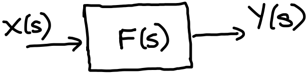
$$p \geq 2 \qquad p \geq \sqrt{2K} \qquad p < 2\sqrt{K}$$



Final design example



Caveat



- Our work on transient response to step input has been for systems with

$$F(s) = F_1(s) = \frac{p_1}{s + p_1}$$

or

$$F(s) = F_2(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Note that they both have a DC Gain of 1.
- What about other systems?

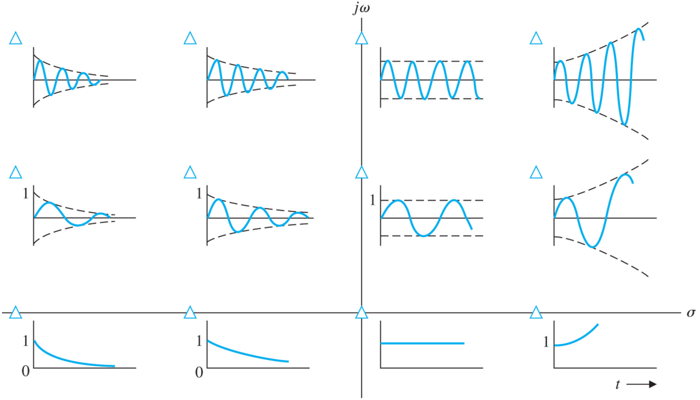
Poles, zeros and transient response

- Consider a general transfer function $F(s) = \frac{Y(s)}{R(s)}$
- Step response: $Y_{\text{step\_resp}}(s) = F(s)\frac{1}{s}$
- Consider case with DC gain = 1; no repeated poles
- Partial fraction expansion

$$Y_{\text{step\_resp}}(s) = \frac{1}{s} + \sum_i \frac{A_i}{s+\sigma_i} + \sum_k \frac{B_k s + C_k}{s^2 + 2\alpha_k s + (\alpha_k^2 + \omega_k^2)}$$

- Step response

$$y_{\text{step\_resp}}(t) = 1 + \sum_i A_i e^{-\sigma_i t} + \sum_k D_k e^{-\alpha_k t} \sin(\omega_k t + \theta_k)$$


jω
σ
1
0
t

Effect of an additional pole

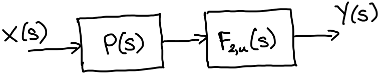
- Let's begin with our second-order under-damped system

where $F(s) = F_{2,u}(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, with $\zeta < 1$.

- Recall, that if $\beta = \sqrt{1-\zeta^2}$ and $\theta = \cos^{-1}(\zeta)$,

$$y_{\text{step\_resp},F_{2,u}}(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

- What if we cascade a system that has a real pole?

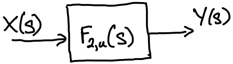


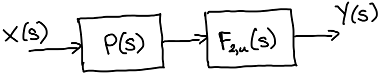
- Now, $Y(s) = P(s)F_{2,u}(s)X(s)$, with $P(s) = \frac{p}{s+p}$
- Step response is now

$$y_{\text{step\_resp},PF_{2,u}}(t) = 1 - Ae^{-\zeta\omega_n t} \sin(\omega_n \beta t + \phi) - Be^{-pt}$$

where A, B, and ϕ are functions of ω_n, ζ and p

Observations





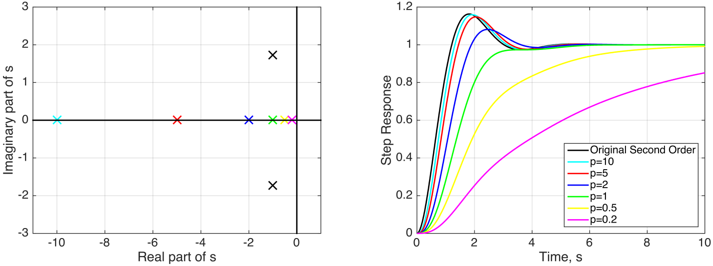
- The step responses are:

$$y_{\text{step\_resp},F_{2,u}}(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$
$$y_{\text{step\_resp},PF_{2,u}}(t) = 1 - A e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \phi) - B e^{-pt}$$

- Observations:
  - If $p \gg \zeta\omega_n$,
    - the extra term decays much faster than the original term
    - Complex poles dominate
  - If p is close to $\zeta\omega_n$, need to consider all poles
  - If $p \ll \zeta\omega_n$,
    - the extra term decays much slower than original terms
    - Begins to resemble a first-order system

Additional pole positions and responses

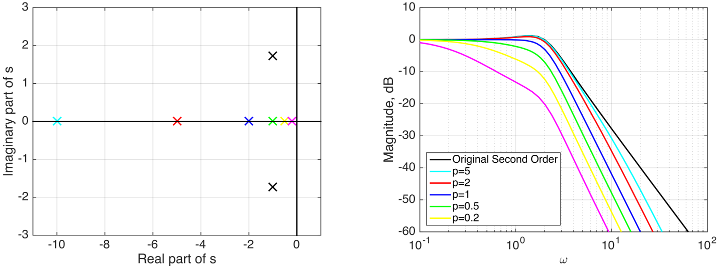
$$Y_{PF_{2,u}}(s) = \left(\frac{p}{s+p}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$



- Why does the new system respond more slowly?
- The additional pole suppresses higher-frequency signals; recall what a pole does to the Bode diagram

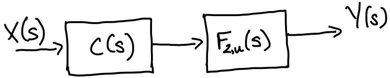
Additional pole Bode diagram

$$Y_{PF_{2,u}}(s) = \left(\frac{p}{s+p}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

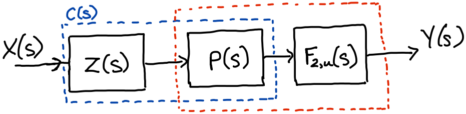


Effect of add. pole and zero

What happens if we also add a zero?



- $Y(s) = C(s)F_{2,u}(s)X(s)$, with $C(s) = \frac{p}{z}\frac{(s+z)}{(s+p)}$.
- For convenience let us redraw

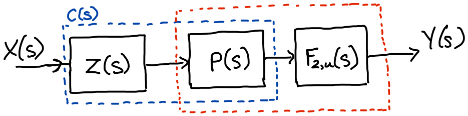


$$Y(s) = Z(s)P(s)F_{2,u}(s)X(s)$$

with $P(s) = \frac{p}{s+p}$ and $Z(s) = \frac{s+z}{z}$.

- Note that $Z(s)$ is not physically realizable in hardware

Analysis



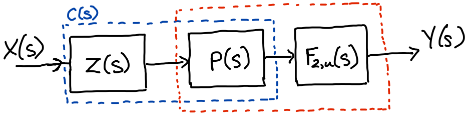
- Note that red box is the "system with an additional pole" that we just considered
- Let $Y_{PF_{2,u}}(s) = P(s)F_{2,u}(s)X(s)$
- Then, recalling that $Z(s) = \frac{s+z}{z}$, we have

$$Y_{CF_{2,u}}(s) = Z(s)Y_{PF_{2,u}}(s) = \frac{1}{z}sY_{PF_{2,u}}(s) + Y_{PF_{2,u}}(s).$$

- That means that

$$y_{\text{step\_resp},CF_{2,u}}(t) = \frac{1}{z}\frac{dy_{\text{step\_resp},PF_{2,u}}(t)}{dt} + y_{\text{step\_resp},PF_{2,u}}(t)$$

Observations



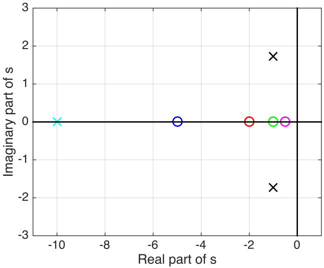
- $y_{\text{step\_resp},PF_{2,u}}(t)$ is the step response of the system with the additional pole; i.e., $P(s)F_{2,u}(s)$
- The step response of the system with the additional pole and zero is

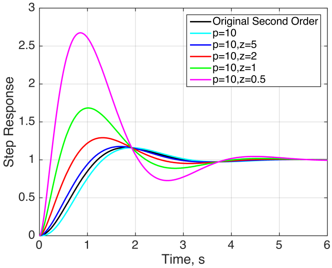
$$y_{\text{step\_resp},CF_{2,u}}(t) = \frac{1}{z}\frac{dy_{\text{step\_resp},PF_{2,u}}(t)}{dt} + y_{\text{step\_resp},PF_{2,u}}(t)$$

- So, if z is big and $y_{\text{step\_resp},PF_{2,u}}(t)$ changes slowly, then $y_{\text{step\_resp},CF_{2,u}}(t)$ will look like $y_{\text{step\_resp},PF_{2,u}}(t)$.
- but speed at which $y_{\text{step\_resp},PF_{2,u}}(t)$ changes is related to the pole positions!

Additional pole and zero positions and responses

$$Y_{CF_{2,u}}(s) = \frac{p}{z}\left(\frac{s+z}{s+p}\right)\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

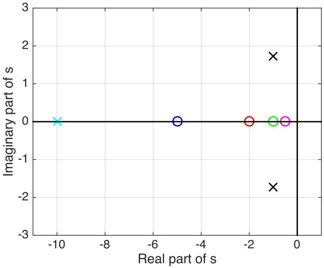


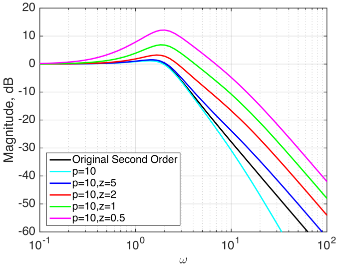


- Why does the new system respond more quickly?
- The additional zero enhances higher-frequency signals; recall what a zero does to the Bode diagram

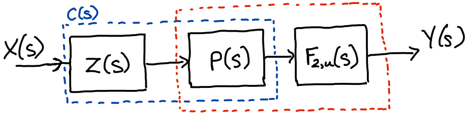
Additional pole and zero Bode diagram

$$Y_{CF_{2,u}}(s) = \frac{p}{z}\left(\frac{s+z}{s+p}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$





Add. pole and non-min.-phase zero



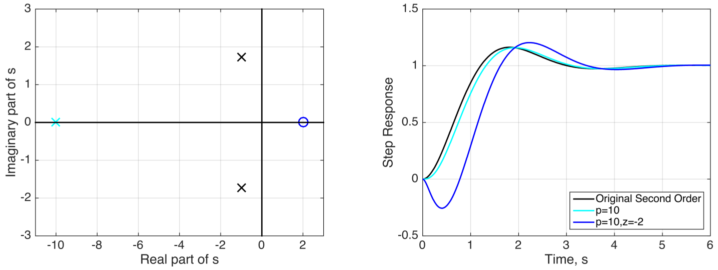
- Recall $Z(s) = \frac{s+z}{z}$
- The step response can be written as:

$$y_{\text{step\_resp},CF_{2,u}}(t) = \frac{1}{z}\frac{dy_{\text{step\_resp},PF_{2,u}}(t)}{dt} + y_{\text{step\_resp},PF_{2,u}}(t)$$

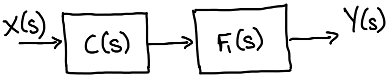
- What happens if we add a zero in the right half plane?
- That is, what happens if z is negative?

Additional pole and non-minimum-phase zero positions and responses

$$Y_{CF_{2,u}}(s) = \frac{p}{z}\left(\frac{s+z}{s+p}\right)\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$



Pole-zero cancellation

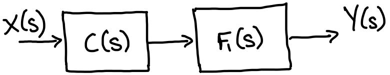


- Cascade of original first order system $F_1(s) = \frac{p_1}{s+p_1}$, and $C(s) = \frac{p}{z}\frac{s+z}{s+p}$
- Transfer function of cascade: $C(s)F_1(s) = \frac{p}{z}\frac{s+z}{s+p}\frac{p_1}{s+p_1}$
- Step response of cascade:

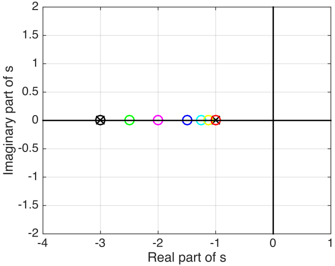
$$y_{\text{step\_resp},CF_1}(t) = 1 - \frac{p(p_1-z)}{z(p_1-p)}e^{-p_1 t} - \frac{p_1(z-p)}{z(p_1-p)}e^{-pt}$$

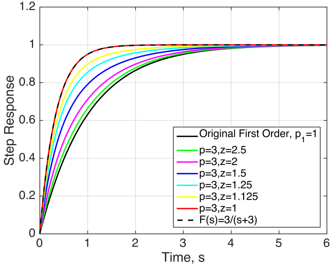
- Looks like we could cancel the dynamics of $F_1(s)$

Pole zero cancellation

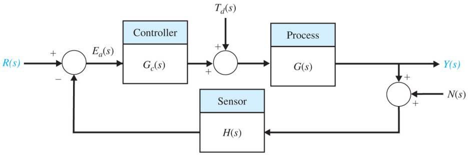


$$C(s)F_1(s) = \frac{p}{z}\left(\frac{s+z}{s+p}\right)\left(\frac{p_1}{s+p_1}\right)$$





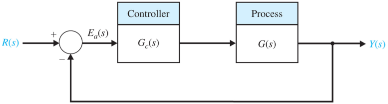
Warnings



(b)

- In control system design, pole-zero cancellation in one transfer function does not necessarily result in pole-zero cancellation in all transfer functions.
- In practice, pole positions are measured and zero positions have to be implemented; subject to measurement and implementation errors
- Hence, care needed when attempting in left half plane
- Never attempt in right half plane

Steady-state error



$$E(s) = R(s) - Y(s) = \frac{1}{1 + G_c(s)G(s)} R(s)$$

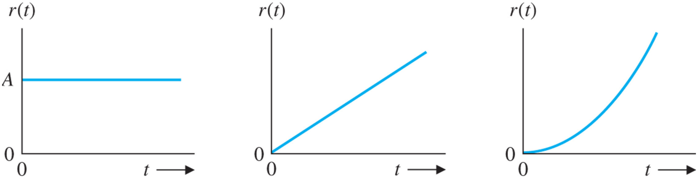
If the the conditions are satisfied, the final value theorem gives steady-state tracking error:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{1}{1 + G_c(s)G(s)} R(s)$$

One of the fundamental reasons for using feedback, despite the cost of the extra components, is to reduce this error.

We will examine this error for the step, ramp and parabolic inputs

Step, ramp, parabolic



(a) (b) (c)

Step input

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{1}{1 + G_c(s)G(s)} R(s)$$

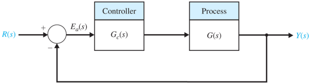
- Step input: $R(s) = \frac{A}{s}$
- $e_{ss} = \lim_{s\to 0} \frac{sA/s}{1+G_c(s)G(s)} = \frac{A}{1+\lim_{s\to 0} G_c(s)G(s)}$
- Now let's examine $G_c(s)G(s)$. Factorize num., den.

$$G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N \prod_{k=1}^{Q}(s+p_k)}$$

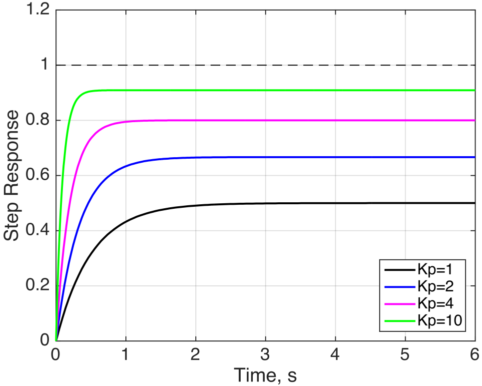
  where $z_i \neq 0$ and $p_k \neq 0$.
- Limit as $s \to 0$ depends strongly on N.
- If $N > 0$, $\lim_{s\to 0} G_c(s)G(s) \to \infty$ and $e_{ss} = 0$
- If $N = 0$,

$$e_{ss} = \frac{A}{1 + G_c(0)G(0)}$$

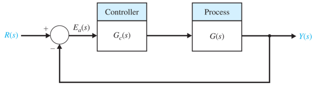
Simple example



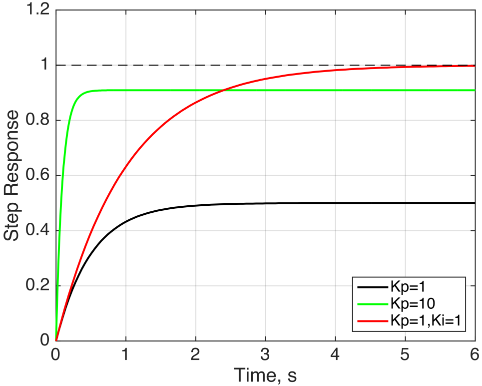
$$G_c(s) = K_p \qquad G(s) = \frac{1}{s+1}$$



Simple example



$$G_c(s) = \frac{K_p s + K_i}{s} \qquad G(s) = \frac{1}{s+1}$$



System types

- Since N plays such a key role, it has been given a name
- It is called the type number
- Hence, for systems of type $N \geq 1$, e_{ss} for a step input is zero
- For systems of type 0, $e_{ss} = \frac{A}{1+G_c(0)G(0)}$

Position error constant

- For type-0 systems, $e_{ss} = \frac{A}{1+G_c(0)G(0)}$
- Sometimes written as $e_{ss} = \frac{A}{1+K_{\text{posn}}}$ where K_{posn} is the position error constant
- Recall $G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}$
- Therefore, for a type-0 system

$$K_{\text{posn}} = \lim_{s\to 0} G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(z_i)}{\prod_{k=1}^{Q}(p_k)}$$

- Note that this can be computed from positions of the non-zero poles and zeros

Ramp input

- The ramp input, which represents a step change in velocity is $r(t) = At$.
- Therefore $R(s) = \frac{A}{s^2}$
- Assuming conditions of final value theorem are satisfied,

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s^2)}{1 + G_c(s)G(s)} = \lim_{s\to 0} \frac{A}{s + sG_c(s)G(s)}$$
$$= \lim_{s\to 0} \frac{A}{sG_c(s)G(s)}$$

- Again, type number will play a key role.

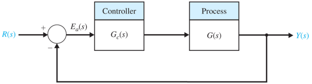
Velocity error constant

- For a ramp input $e_{ss} = \lim_{s\to 0} \frac{A}{sG_c(s)G(s)}$
- Recall $G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}$
- For type-0 systems, $G_c(s)G(s)$ has no poles at origin. Hence, $e_{ss} \to \infty$
- For type-1 systems, $G_c(s)G(s)$ has one pole at the origin. Hence, $e_{ss} = \frac{A}{K_v}$, where $K_v = \frac{K\prod_i z_i}{\prod_k p_k}$
- Note K_v can be computed from non-zero poles and zeros
- Suggests formal definition of velocity error constant

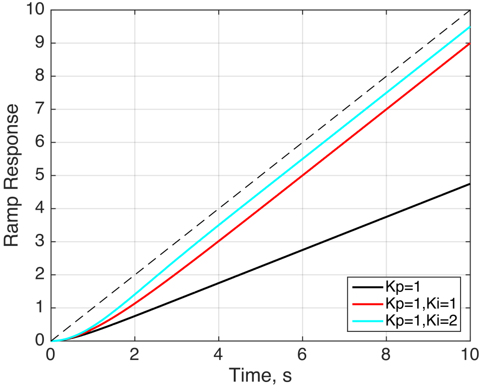
$$K_v = \lim_{s\to 0} sG_c(s)G(s)$$

- For type-N systems with $N \geq 2$, for a ramp input $e_{ss} = 0$

Simple example



$$G_c(s) = \frac{K_p s + K_i}{s} \qquad G(s) = \frac{1}{s+1}$$



Parabolic input

- The parabolic input, which represents a step change in acceleration is $r(t) = At^2/2$.
- Therefore $R(s) = \frac{A}{s^3}$
- Assuming conditions of final value theorem are satisfied,

$$e_{ss} = \lim_{s \to 0} \frac{s(A/s^3)}{1 + G_c(s)G(s)} = \lim_{s \to 0} \frac{A}{s^2 G_c(s)G(s)}$$

- Again, type number will play a key role.

Acceleration error constant

- For a parabolic input $e_{ss} = \lim_{s\to 0} \frac{A}{s^2 G_c(s)G(s)}$
- Recall $G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N \prod_{k=1}^{Q}(s+p_k)}$
- For type-0 and type-1 systems, $G_c(s)G(s)$ has at most one pole at origin. Hence, $e_{ss} \to \infty$
- For type-2 systems, $G_c(s)G(s)$ has two poles at the origin. Hence, $e_{ss} = \frac{A}{K_a}$, where $K_a = \frac{K\prod_i z_i}{\prod_k p_k}$
- Again, K_a can be computed from non-zero poles and zeros
- Suggests formal definition of acceleration error constant

$$K_a = \lim_{s\to 0} s^2 G_c(s)G(s)$$

- For type-N systems with $N \geq 3$, for a parabolic input $e_{ss} = 0$

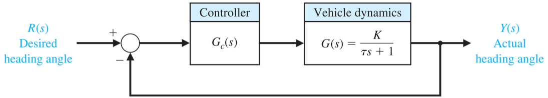
Summary of steady-state errors

Table 5.5 Summary of Steady-State Errors

Number of Integrations in $G_c(s)G(s)$, Type Number	Input: Step, $r(t) = A$, $R(s) = A/s$	Input: Ramp, At, A/s^2	Input: Parabola, $At^2/2$, A/s^3
0	$e_{ss} = \frac{A}{1 + K_p}$	Infinite	Infinite
1	$e_{ss} = 0$	$\frac{A}{K_v}$	Infinite
2	$e_{ss} = 0$	0	$\frac{A}{K_a}$

The K_p in this table corresponds to K_{posn}

Robot steering system, P control



Let's examine a proportional controller:

$$G_c(s) = K_1$$

- $G_c(s)G(s) = K_1 K/(\tau s + 1)$
- Hence, $G_c(s)G(s)$ is a type-0 system.
- Hence, for a step input,

$$e_{ss} = \frac{A}{1 + K_{\text{posn}}}$$

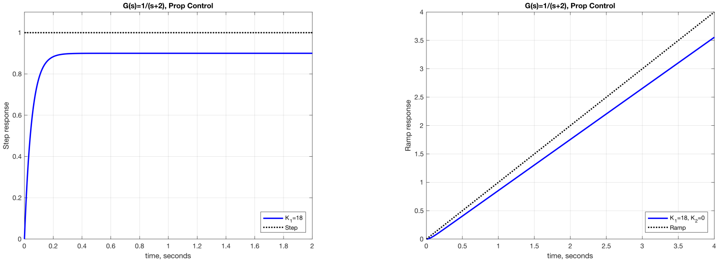
where $K_{\text{posn}} = K_1 K$.

Robot steering system, P control example

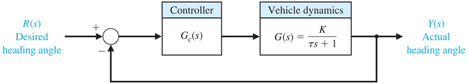
- Let $G(s) = \frac{1}{s+2} = \frac{0.5}{0.5s+1}$.
- Proportional control, $G_c(s) = K_1$. Choose $K_1 = 18$.
- Since $G_c(s)G(s)$ is type-0:
  - finite steady-state error for a step,
  - unbounded steady-state error for a ramp
- In this example, $K_{\text{posn}} = KK_1 = 9$
- The steady-state error for a step input will be $\frac{1}{1+K_{\text{posn}}} = 10\%$ of the height of the step.
- For a unit step the steady-state error will be 0.1.

Robot steering system, P control example

- Left: $y(t)$ for unit step input, $r(t) = u(t)$
- Right: $y(t)$ for unit ramp input, $r(t) = tu(t)$



Robot steering system, PI control



Let's examine a proportional-plus-integral controller:

$$G_c(s) = K_1 + \frac{K_2}{s} = \frac{K_1 s + K_2}{s}$$

- When $K_2 \neq 0$, $G_c(s)G(s) = \frac{K(K_1 s + K_2)}{s(\tau s + 1)}$
- Hence, $G_c(s)G(s)$ is a type-1 system.
- Hence, for a step input, $e_{ss} = 0$
- For ramp input,

$$e_{ss} = \frac{A}{K_v},$$

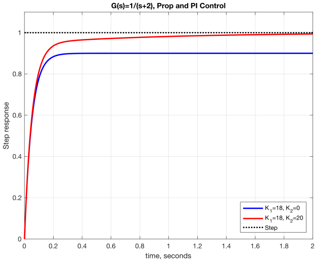
where $K_v = \lim_{s \to 0} sG_c(s)G(s) = KK_2$

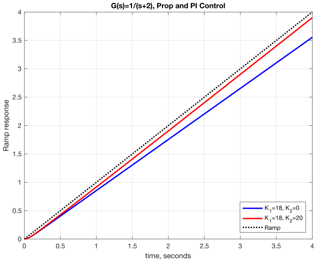
Robot steering system, PI control example

- Same system: $G(s) = \frac{1}{s+2} = \frac{0.5}{0.5s+1}$.
- Prop. + Int. control, $G_c(s) = K_1 + \frac{K_2}{s} = \frac{K_1 s + K_2}{s}$. Choose $K_1 = 18$ and $K_2 = 20$.
- Now since $G_c(s)G(s)$ is type-1:
  - zero steady-state error for a step
  - finite-steady state error for a ramp
- In this example $K_v = KK_2 = 10$
- The steady-state error for a ramp input will be $\frac{1}{K_v} = 10\%$ of the slope of the ramp.
- For a unit ramp the steady-state error will be 0.1.

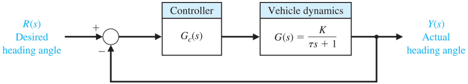
Robot steering system, PI control example

- Left: $y(t)$ for unit step input, $r(t) = u(t)$
- Right: $y(t)$ for unit ramp input, $r(t) = tu(t)$





Robot steering system, PI2I control



Let's examine a PI plus double integral controller:

$$G_c(s) = K_1 + \frac{K_2}{s} + \frac{K_3}{s^2} = \frac{K_1 s^2 + K_2 s + K_3}{s^2}$$

- When $K_3 \neq 0$, $G_c(s)G(s) = \frac{K(K_1 s^2 + K_2 s + K_3)}{s^2(\tau s + 1)}$
- Hence, $G_c(s)G(s)$ is a type-2 system.
- Hence, for a step input or a ramp input, $e_{ss} = 0$
- For parabolic input,

$$e_{ss} = \frac{A}{K_a},$$

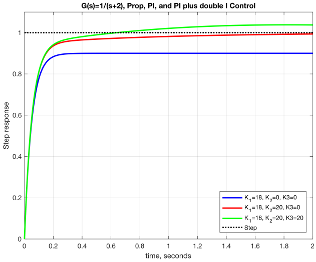
where $K_a = \lim_{s \to 0} s^2 G_c(s)G(s) = KK_3$

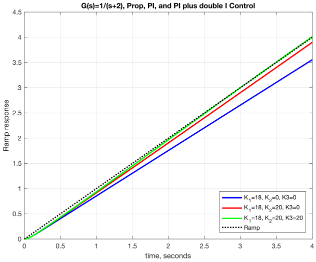
Robot steering system, PI2I control example

- Same system: $G(s) = \frac{1}{s+2} = \frac{0.5}{0.5s+1}$.
- Prop. + Int. + double int. control, $G_c(s) = K_1 + \frac{K_2}{s} + \frac{K_3}{s^2}$. Choose $K_1 = 18$, $K_2 = 20$, $K_3 = 20$.
- Now since $G_c(s)G(s)$ is type-2:
  - zero steady-state error for a step or a ramp
  - finite-steady state error for a parabolic
- In this example $K_a = KK_3 = 10$
- The steady-state error for a parabolic input would be $\frac{1}{K_v} = 10\%$ of the curvature of the parabola.
- For a unit parabola the steady-state error would be 0.1.

Robot steering system, PI2I control example

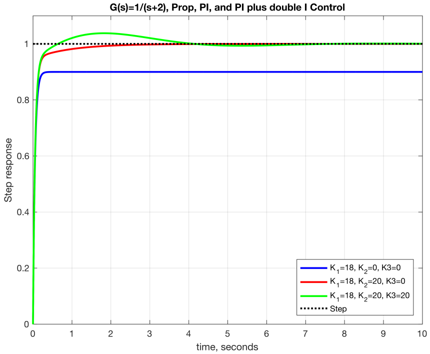
- Left: $y(t)$ for unit step input, $r(t) = u(t)$
- Right: $y(t)$ for unit ramp input, $r(t) = tu(t)$





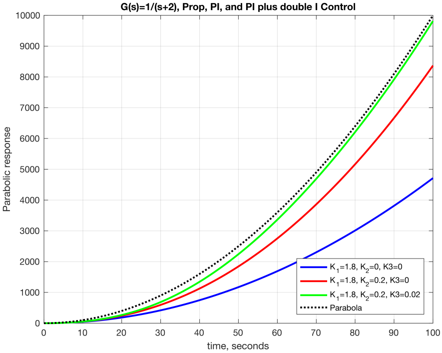
Robot steering system, PI2I control example

- $y(t)$ for unit step input, $r(t) = u(t)$, extended time scale



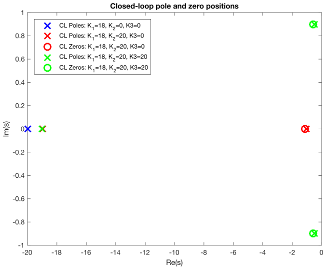
Robot steering system, PI2I control example

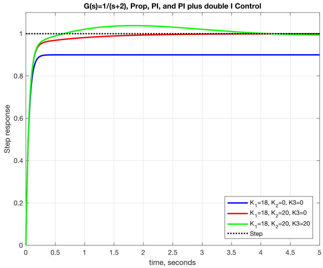
- $y(t)$ for unit parabolic input, $r(t) = t^2u(t)$
- For this slide only, the gains have been reduced to illustrate the effects, $K_1 = 1.8$, $K_2 = 0.2$, $K_3 = 0.02$



Transient responses and poles

Should we have been able to predict transient responses from pole (and zero) positions? Return to case of $K_1 = 18, K_2 = K_3 = 20$





Closed loop transfer functions, $T(s) = \frac{Y(s)}{R(s)}$:

- **P** one real pole, time const. = 1/20 = 0.05s
- **PI** one real pole near the P one; plus another real pole (time const. $\approx$ 1s) that is close to a zero
- **PI2I** one real pole near the P one; plus a conjugate pair with time const. $\approx$ 2s, angle $\approx 60^\circ$, but near zeros

Step responses

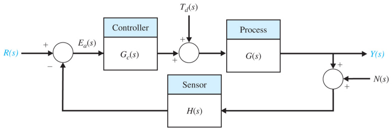
To highlight the impacts of the different poles, we have done a partial fraction expansion of the transfer function and used that to compute the step response

Control	$T(s) = \frac{Y(s)}{R(s)}$	**Step Response**, for $t \geq 0$
P	$= \frac{18}{s+20}$	$= 0.9 - 0.9e^{-20t}$
PI	$= \frac{18s+20}{s^2+20s+20}$ $\simeq \frac{17.94}{s+18.94} + \frac{0.0557}{s+1.056}$	$\simeq 1 - 0.947e^{-18.94t} - 0.053e^{-1.056t}$
PI2I	$= \frac{18s^2+20s+20}{s^3+20s^2+20s+20}$ $\simeq \frac{17.89}{s+19.00} + \frac{0.1106(s+0.5578)}{s^2+0.9971s+1.0525}$	$\simeq 1 - 0.942e^{-19.00t} \ldots$ $-0.108e^{-0.498t}\sin(0.897t + 2.57)$

Notes:

- 10% steady state error in the P case; it is zero in other cases
- Second term for each system has a similar decay rate (similar pole positions)
- Third term in PI case decays much more slowly; third term in PI2I case even slower (small real parts of these poles)
- Terms related to poles that are near zeros have comparatively small magnitudes

Summary: Desirable properties



(b)

With $H(s) = 1$, $E(s) = R(s) - Y(s)$, $L(s) = G_c(s)G(s)$,

$$E(s) = \frac{1}{1 + L(s)} R(s) - \frac{G(s)}{1 + L(s)} T_d(s) + \frac{L(s)}{1 + L(s)} N(s)$$

- Stability
- Good tracking in the steady state
- Good tracking in the transient
- Good disturbance rejection (good regulation)
- Good noise suppression
- Robustness to model mismatch (discussed later in course)

Plan: Analysis and design techniques

Rest of course: about developing analysis and design techniques to address these goals

- Routh-Hurwitz:
  - Enables us to determine stability without having to find the poles of the denominator of a transfer function
- Root locus
  - Enables us to show how the poles move as a single design parameter (such as an amplifier gain) changes
- Bode diagrams
  - There is often enough information in the Bode diagram of the plant/process to construct a highly effective design technique
- Nyquist diagram
  - More advanced analysis of the frequency response that enables stability to be assessed even for complicated systems