EE3CL4: Introduction to Linear Control Systems

Section 9: Design of Lead and Lag Compensators using Frequency Domain Techniques

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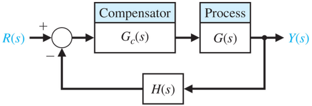
Outline

Frequency domain design



- Analyze closed loop using open loop transfer function $L(s) = G_c(s)G(s)H(s)$.
- We would like closed loop to be stable:
  - Use Nyquist's stability criterion (on $L(s)$)
- We might like to make sure that the closed loop remains stable even if there is an increase in the gain
  - Require a particular gain margin (of $L(s)$)
- We might like to make sure that the closed loop remains stable even if there is additional phase lag
  - Require a particular phase margin (of $L(s)$)
- We might like to make sure that the closed loop remains stable even if there is a combination of increased gain and additional phase lag

Robust stability

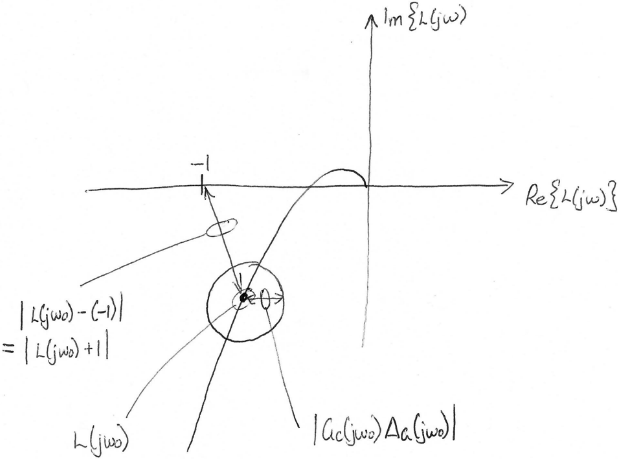
- Let $\breve{G}(s)$ denote the true plant and let $G(s)$ denote our model
- $\Delta_G(s) = \breve{G}(s) - G(s)$ denotes the uncertainty in our model
- If $\breve{G}(s)$ has the same number of RHP poles as $G(s)$, we need to ensure that the Nyquist plot of

$$\breve{L}(s) = G_c(s)\breve{G}(s) = L(s) + G_c(s)\Delta_G(s)$$

has the same number of encirclements of -1 as the plot of $L(s)$.

- This will give us a sufficient condition for robust stability

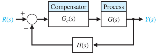
Robust stability II



Robust stability III

- Our sufficient condition is $|1 + L(j\omega)| > |G_c(j\omega)\Delta_G(j\omega)|$.
- That is equivalent to $|\frac{1}{L(j\omega)} + 1| > \left|\frac{\Delta_G(j\omega)}{G(j\omega)}\right|$
- That is, we need $|L(j\omega)|$ to be small at the frequencies where the relative error in our model is large; typically at higher frequencies

Frequency domain design



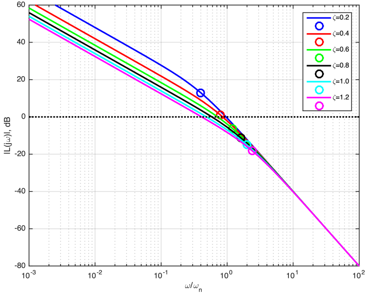
- We might like to control the damping ratio of the dominant pole pair
  - Use the fact that $\phi_{pm} = f(\zeta)$;
- We might like to control the steady-state error constants
  - For step, ramp and parabolic inputs, these constants are related to the behaviour of $L(s)$ around zero; i.e., behaviour near DC. Recall $K_{\text{posn}} = L(0)$ and $K_v = \lim_{s\to 0} sL(s)$.
- We might like to influence the settling time
  - Roughly speaking, the settling time decreases with increasing closed-loop bandwidth. How is this related to bandwidth of $L(s)$?

Bandwidth

- Let ω_c be the (open-loop) cross-over frequency; i.e., $|L(j\omega_c)| = 1$
- Let $T(s) = \frac{Y(s)}{R(s)} = \frac{L(s)}{1+L(s)}$.
- Consider a low-pass open loop transfer function
- When $\omega \ll \omega_c$, $|L(j\omega)| \gg 1, \implies T(j\omega) \approx 1$
- When $\omega \gg \omega_c$, $|L(j\omega)| \ll 1, \implies T(j\omega) \approx L(j\omega)$
- Can we quantify things a bit more, and perhaps gain some insight, for a standard second-order system

Bandwidth, open loop

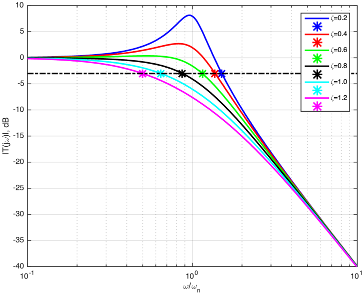
- For a standard second-order system, $L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$
- To sketch open loop Bode diagram, $L(j\omega) = \frac{\omega_n/(2\zeta)}{j\omega\left(1+j\omega/(2\zeta\omega_n)\right)}$
- Low freq's: slope of -20 dB/decade; Corner freq. at $2\zeta\omega_n$;
  High freq's: slope of -40dB/decade
- Crossover frequency: $\omega_c = \omega_n\left(\sqrt{1+4\zeta^4} - 2\zeta^2\right)^{1/2}$



Circles are the corner frequencies; Observe crossover frequencies

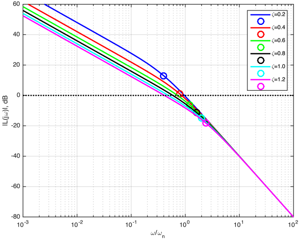
Bandwith, closed loop

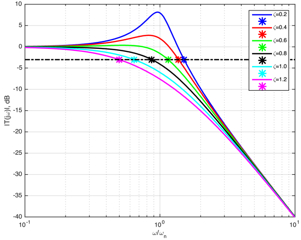
- To sketch closed-loop Bode diagram, $T(j\omega) = \frac{1}{1+j2\zeta\omega/\omega_n-(\omega/\omega_n)^2}$
- Low freq's: slope of zero; Double corner frequency at ω_n; High freq's: slope of -40dB/decade
- For $\zeta < 1/\sqrt{2}$, peak of $\frac{1}{2\zeta\sqrt{1-\zeta^2}}$ at $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ (Lab 2)
- 3dB bandwidth: $\omega_B = \omega_n\left(\sqrt{2-4\zeta^2+4\zeta^4}+1-2\zeta^2\right)^{1/2}$, $\approx \omega_n(-1.19\zeta+1.85)$ for $0.3 \leq \zeta \leq 0.8$.



Asterisks are ω_B

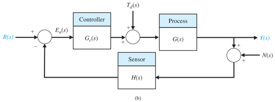
Bandwidth, open and closed loops





- OL crossover freq.: $\omega_c = \omega_n\left(\sqrt{1+4\zeta^4} - 2\zeta^2\right)^{1/2}$
- CL 3dB BW: $\omega_B = \omega_n\left(\sqrt{2-4\zeta^2+4\zeta^4} + 1 - 2\zeta^2\right)^{1/2}$
- 2% settling time: $T_{s,2} \approx \frac{4}{\zeta\omega_n}$
- Rise time (0% $\to$ 100%) of step response: $\frac{\pi/2+\sin^{-1}(\zeta)}{\omega_n\sqrt{1-\zeta^2}}$
- Close relationship with ω_c and ω_B, esp. through ω_n. Care needed in dealing with damping effects.

Loopshaping, again



$$E(s) = \frac{1}{1+L(s)}\,R(s) - \frac{G(s)}{1+L(s)}\,T_d(s) + \frac{L(s)}{1+L(s)}\,N(s)$$

where, with $H(s) = 1$, $L(s) = G_c(s)G(s)$

What design insights are available in the frequency domain?

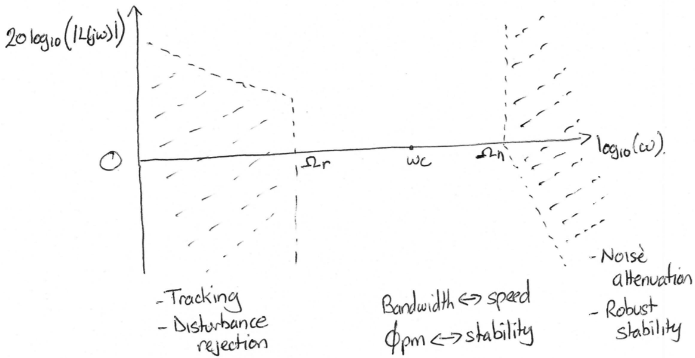
- Good tracking: $\implies$ $L(s)$ large where $R(s)$ large
  $|L(j\omega)|$ large in the important frequency bands of $r(t)$
- Good dist. rejection: $\implies$ $L(s)$ large where $T_d(s)$ large
  $|L(j\omega)|$ large in the important frequency bands of $t_d(t)$
- Good noise suppr.: $\implies$ $L(s)$ small where $N(s)$ large
  $|L(j\omega)|$ small in the important frequency bands of $n(t)$
- Robust stability: $\implies$ $L(s)$ small where $\frac{\Delta_G(s)}{G(s)}$ large
  $|L(j\omega)|$ small in freq. bands where relative error in model large
- Phase margin: $\angle L(j\omega)$ away from -180° when $|L(j\omega)|$ close to 1

Typically, $L(j\omega)$ is a low-pass function,

How can we visualize these things?

- Interesting properties of $L(s)$: encirclements, gain margin, phase margin, general stability margin, gain at low frequencies, bandwidth (ω_c), gain at high frequencies, phase around the cross-over frequency
- All this information is available from the Nyquist diagram
- Not always easily accessible
- Once we have a general idea of the shape of the Nyquist diagram, is some of this information available in a more convenient form? at least for relatively simple systems?

Bode diagram



Seems to capture most issues, but

How fast can we transition from high open-loop gain to low open-loop gain?

This is magnitude. What can we say about phase?

Phase from magnitude?

- For systems with more poles than zeros and all the poles and zeros in the left half plane, we can write a formal relationship between gain and phase. That relationship is a little complicated, but we can gain insight through a simplification.
- Assume that ω_c is some distance from any of the corner frequencies of the open-loop transfer function. That means that around ω_c, the Bode magnitude diagram is nearly a straight line
- Let the slope of that line be $-20n$ dB/decade
- Then for these frequencies $L(j\omega) \approx \frac{K}{(j\omega)^n}$
- That means that for these frequencies $\angle L(j\omega) \approx -n90^\circ$
- That suggests that at the crossover frequency the Bode magnitude plot should have a slope around -20dB/decade in order to have a good phase margin
- For more complicated systems we need more sophisticated results, but the insight of shallow slope of the magnitude diagram around the crossover frequency applies for large classes of practical systems

Compensators and Bode diagram

- We have seen the importance of phase margin
- If $G(s)$ does not have the desired margin, how should we choose $G_c(s)$ so that $L(s) = G_c(s)G(s)$ does?
- To begin, how does $G_c(s)$ affect the Bode diagram
- Magnitude:

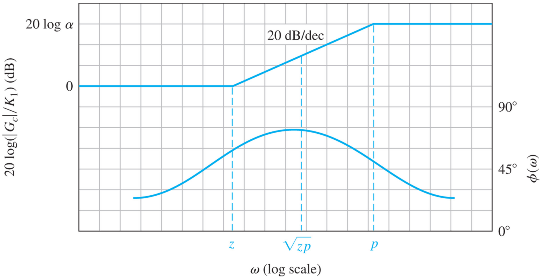
$$
\begin{aligned}
&20\log_{10}\big(|G_c(j\omega)G(j\omega)|\big) \\
&\qquad = 20\log_{10}\big((|G_c(j\omega)|\big) + 20\log_{10}\big(|G(j\omega)|\big)
\end{aligned}
$$

- Phase:

$$\angle G_c(j\omega)G(j\omega) = \angle G_c(j\omega) + \angle G(j\omega)$$

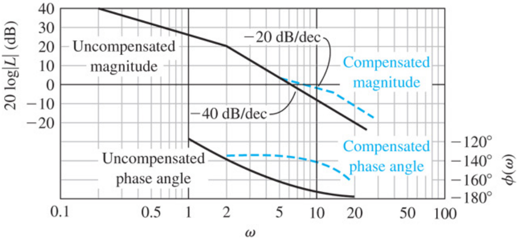
Lead Compensators

- $G_c(s) = \frac{K_c(s+z)}{s+p}$, with $|z| < |p|$, alternatively,
- $G_c(s) = \frac{K_c}{\alpha}\frac{1+s\alpha_{\text{lead}}\tau}{1+s\tau}$, where $p = 1/\tau$ and $\alpha_{\text{lead}} = p/z > 1$
- Bode diagram (in the figure, $K_1 = K_c/\alpha_{\text{lead}}$):

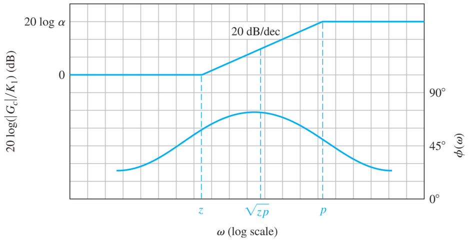


Lead Compensation

- What will lead compensation, do?
- Phase is positive: might be able to increase phase margin ϕ_{pm}
- Slope is positive: might be able to increase the cross-over frequency, ω_c, (and the bandwidth)



Lead Compensation



- $G_c(s) = \frac{K_c}{\alpha_{\text{lead}}} \frac{1+s\alpha_{\text{lead}}\tau}{1+s\tau}$
- By making the denom. real, can show that $\angle G_c(j\omega) = \text{atan}\left(\frac{\omega\tau(\alpha_{\text{lead}}-1)}{1+\alpha_{\text{lead}}(\omega\tau)^2}\right)$
- Max. occurs when $\omega = \omega_m = \frac{1}{\tau\sqrt{\alpha_{\text{lead}}}} = \sqrt{zp}$
- Max. phase angle satisfies $\tan(\phi_m) = \frac{\alpha_{\text{lead}}-1}{2\sqrt{\alpha_{\text{lead}}}}$
- Equivalently, $\sin(\phi_m) = \frac{\alpha_{\text{lead}}-1}{\alpha_{\text{lead}}+1}$
- At $\omega = \omega_m$, we have $|G_c(j\omega_m)| = K_c/\sqrt{\alpha_{\text{lead}}}$

Bode Design Principles (lead)

- Select the desired (open loop) crossover frequency and the desired phase margin based on loop shaping ideas and the desired transient response
- Set the amplifier gain so that proportionally controlled open loop has a gain of 1 at chosen crossover frequency
- Evaluate the phase margin
- If the phase marking is insufficient, use the phase lead characteristic of the lead compensator $G_c(s) = K_c \frac{s+z}{s+p}$ with $p = \alpha_{\text{lead}} z$ and $\alpha_{\text{lead}} > 1$ to improve this margin
  - Do this by placing the peak of the phase of the lead compensator at ω_c and by ensuring that the value of the peak is large enough for $\angle L(j\omega_c)$ to meet the phase margin specification. That will give you z and p
  - Choose K_c so that the loop gain at ω_c is still one; i.e., $|L(j\omega_c)| = 1$
- Evaluate other performance criteria

Bode Design Practice (lead)

- If the phase margin is insufficient, use the phase lead characteristic of the lead compensator $G_c(s) = K_c \frac{s+z}{s+p}$ with $p = \alpha_{\text{lead}} z$ and $\alpha_{\text{lead}} > 1$ to improve this margin
  - Determine the additional phase lead required ϕ_{add}
  - Provide this additional phase lead with the peak phase of the lead compensator; that is, choose $\alpha_{\text{lead}} = \frac{1+\sin(\phi_{\text{add}})}{1-\sin(\phi_{\text{add}})}$
  - Place that peak of phase at the desired value of ω_c; that is, select z and p with $p = \alpha_{\text{lead}} z$ such that $\sqrt{zp} = \omega_c$.
  - Set K_c such that $K_c \left| \frac{j\omega_c+z}{j\omega_c+p} G(j\omega_c) \right| = 1$.
- Evaluate other performance criteria

Example, Lead

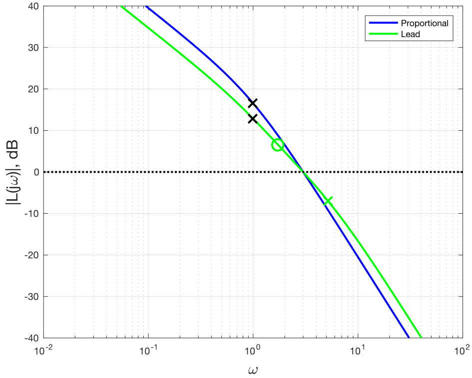
- Type 1 plant of order 2: $G(s) = \frac{0.2}{s(s+1)}$
- Design goals:
  - Open loop crossover frequency at $\omega_c \approx 3\text{rads}^{-1}$.
  - Phase margin of 45° (implies a damping ratio)

- Try to achieve this with proportional control.
- $|G(j3)| = \frac{0.2}{3\sqrt{10}}$.
- To make $L(j3) = 1$ with a proportional controller we choose $K_{\text{amp}} = 15\sqrt{10}$
- In that case,
  $\phi_{pm} = 180 + \angle G(j\omega_c) = 180^\circ - 90^\circ - \arctan(3) \approx 18^\circ$
- Fails to meet specifications

Lead compensator design

- Use a lead controller of the form $G_c(s) = K_c \frac{s+z}{s+p}$
- Need to add at least $\phi_{\text{add}} = 27^\circ$ of phase at $\omega_c = 3\text{rads}^{-1}$
  Let's add $\phi_{\text{add}} = 30^\circ$, to account for imperfect implementation
- Determine α_{lead} using $\alpha_{\text{lead}} = \frac{1+\sin(\phi_{\text{add}})}{1-\sin(\phi_{\text{add}})} = 3$. Thus, $p = 3z$.
- Need to put this phase at $\omega_c = 3\text{rads}^{-1}$.
  Thus need $\sqrt{zp} = \sqrt{3z^2} = 3$.
  Therefore, $z = \sqrt{3} \approx 1.73$; $p = 3\sqrt{3} \approx 5.20$.
- Choose K_c such that with $\omega_c = 3$, $\left|K_c \frac{j\omega_c+1.73}{j\omega_c+5.20} \frac{0.2}{j\omega_c(j\omega_c+1)}\right| = 1$
- Thus $K_c \approx 82.2$.
- Thus lead controller is $G_c(s) = 82.2 \frac{s+1.73}{s+5.20}$.
- Resulting crossover frequency is indeed $\omega_c = 3$;
  phase margin is $\phi_{pm} = 48.5^\circ$.

Bode Mag Diagrams, open loop

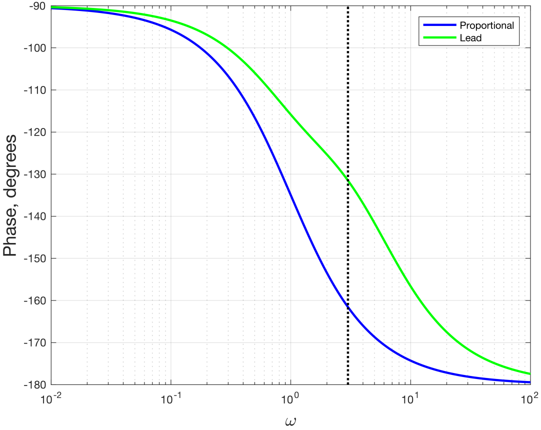


Black x: marks frequency of plant pole;

Green x and circle: frequencies of lead compensator pole and zero

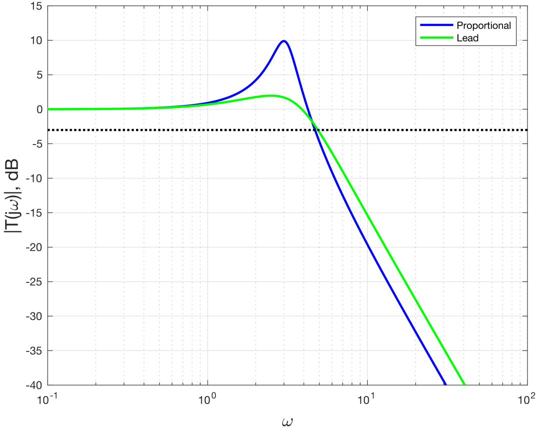
Same cross over frequency; lead has shallower slope

Bode Phase Diagrams, open loop



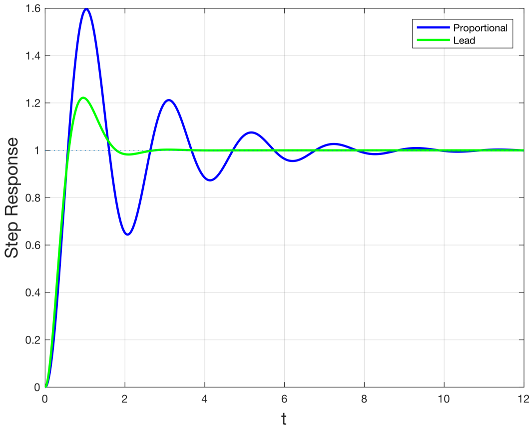
Observe additional phase from lead compensator and improved phase margin

Bode Mag Diagrams, closed loop



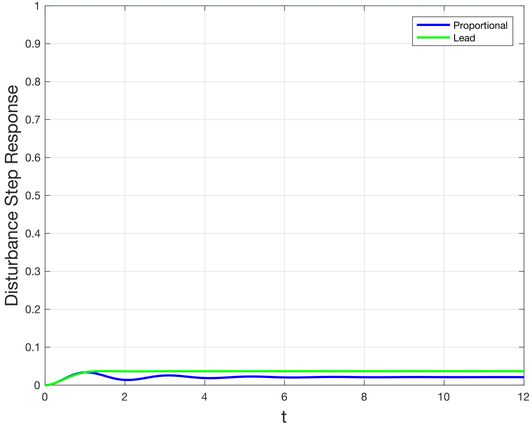
Note reduction in resonant peak (reflects larger damping ratio)

Step Responses



Note reduction in overshoot (larger damping ratio), and shorter settling time (wider closed-loop bandwidth)

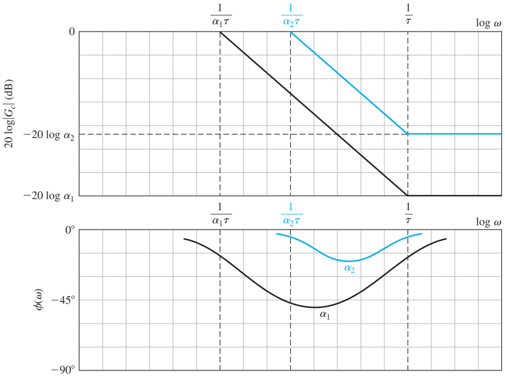
Responses to step disturbance



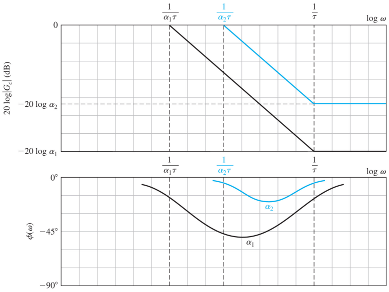
Disturbance response of lead design is worse due to smaller low-freq. open loop gain

Lag Compensators

- $G_c(s) = \frac{K_c(s+z)}{s+p}$, with $|p| < |z|$, alternatively,
- $G_c(s) = \frac{K_c\alpha_{\text{lag}}(1+s\tau)}{1+s\alpha_{\text{lag}}\tau}$, where $z = 1/\tau$ and $\alpha_{\text{lag}} = z/p > 1$
- Low frequency gain: $K_c\frac{z}{p} = K_c\alpha_{\text{lag}}$.
- High frequency Gain: K_c
- Bode diagrams of lag compensators for two different α_{lag}s, in the case where $K_c = 1/\alpha_{\text{lag}}$



What will lag compensation do?



- Larger gains at lower frequencies; have the potential to improve steady-state error constants for step and ramp, and to provide better rejection of low-frequency disturbances
- However, phase lag characteristic could reduce phase margin
- Address this by ensuring that position of the zero is well below the crossover frequency. That way the phase lag added at ω_c will be small.

Bode Design Principles (lag)

For lag compensators:

- Add gain at low frequencies to improve steady state error constants and low-frequency disturbance rejection without changing (very much) the crossover frequency nor the phase margin

Design Guidelines

1. Select the desired (open loop) crossover frequency and the desired phase margin based on loop shaping ideas and the desired transient response.
2. Select the desired steady-state error coefficients
3. For uncompensated (i.e., proportionally controlled) closed loop, set amplifier gain K_{amp} so that open loop crossover frequency is in the desired position
4. Check that this uncompensated system achieves the desired phase margin. If not, stop. We will need to lead compensate the plant first.
5. If the specified phase margin is achieved, proceed with the design of lag compensator $G_c(s) = \frac{K_c(s+z)}{s+p}$.

Design Guidelines, cont.

6. Determine factor by which low-frequency gain needs to be increased. This factor is α_{lag}
7. Set the zero z so that it is factor of around 30 below the crossover frequency to ensure that phase lag added by lag compensator at that frequency is small.
8. Set the pole $p = z/\alpha_{\text{lag}}$.
9. Set $K_c = K_{\text{amp}}$.

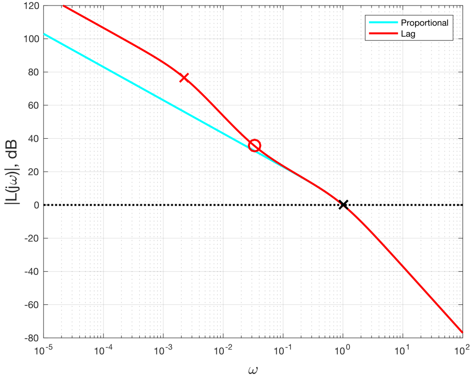
Example, lag

- Type 1 plant of order 2: $G(s) = \frac{0.2}{s(s+1)}$
- Design goals:
  - Open loop crossover frequency at $\omega_c = 1$rads^{-1} (recall lead design had $\omega_c = 3$)
  - Phase margin at least 45°
  - Velocity error constant of $K_v = 20$.
- See if we can achieve this using proportional control.
- To achieve $\left|K_{\text{amp}} G(j1)\right| = 1$ we choose $K_{\text{amp}} = 10/\sqrt{2}$.
- $\angle G(j1)/\sqrt{2} = -135°$. Hence, phase margin criterion is satisfied.
- With $K_{\text{amp}} = 10/\sqrt{2}$, $K_v = \lim_{s\to 0} sK_{\text{amp}} G(s) = \sqrt{2}$.
- Fails to meet specification

Example

- To meet the requirement on K_v we need to increase low-frequency gain by $\alpha_{\text{lag}} = 20/\sqrt{2} \lesssim 15$
- To ensure that lag compensator does not reduce phase margin (by very much), set $z = \frac{\omega_c}{30} = \frac{1}{30}$
- Set $p = z/\alpha_{\text{lag}} = \frac{1}{450}$.
- Set $K_c = K_{\text{amp}} = 10\sqrt{2}$
- Hence lag controller is $G_c(s) = \frac{7.07(s+1/30)}{s+1/450}$.

Bode Mag Diagrams, open loop

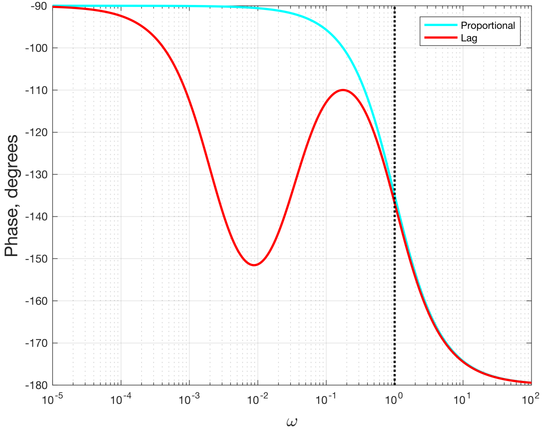


Black x: frequency of plant pole;

Red x and circle: frequencies of lag compensator pole and zero

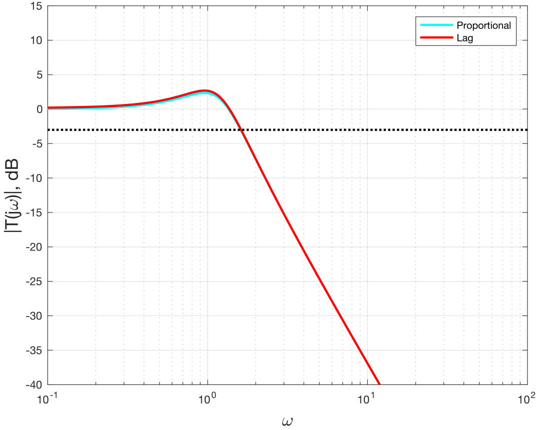
Same cross over frequency; lag has larger low-frequency open-loop gain

Bode Phase Diagrams, open loop



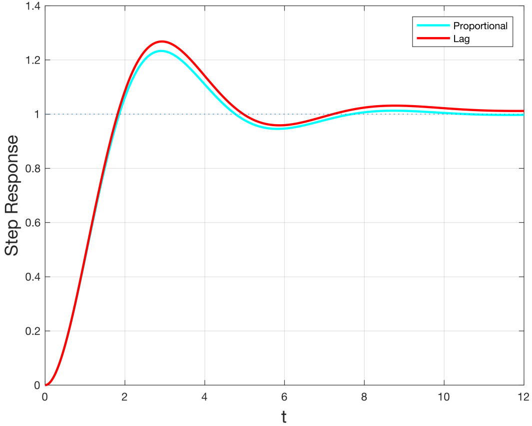
Observe additional phase lag from compensator but that it is very small near crossover frequency

Bode Mag Diagrams, closed loop



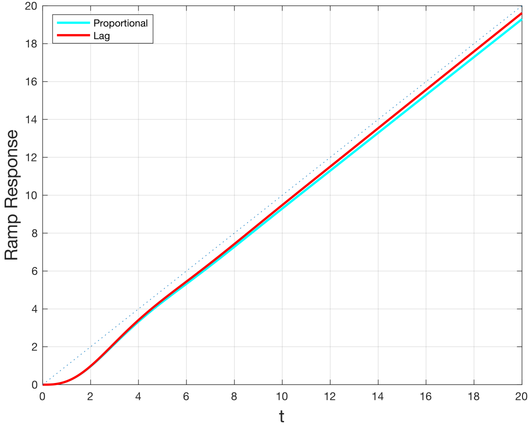
Note similar closed loop frequency response (as we would expect from design)

Step Responses



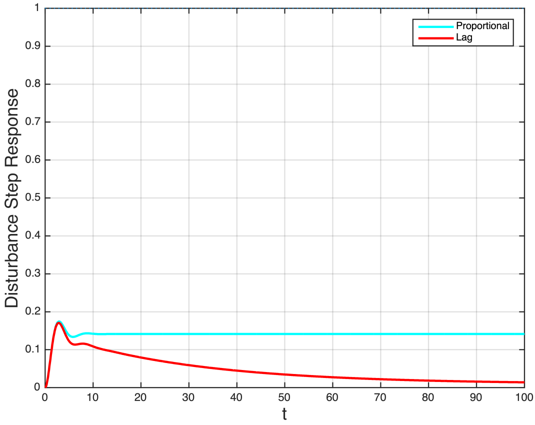
Similar, by design

Ramp Responses



Lag has reduced steady-state error, by design

Responses to step disturbance



Larger low-frequency open-loop gain of lag design yields better step disturbance rejection

Lead-lag design

- If the design specifications include
  - crossover frequency
  - phase margin
  - steady-state error constants or low frequency disturbance rejection
- Then
  - If first two goals cannot be achieved using proportional control, design a phase-lead compensator for $G(s)$ to achieve them, then
  - Design a phase-lag compensator for $\tilde{G}(s) = G_{c,\text{lead}}(s)G(s)$ to increase the low-frequency gain without changing (very much) the crossover frequency nor the phase margin.

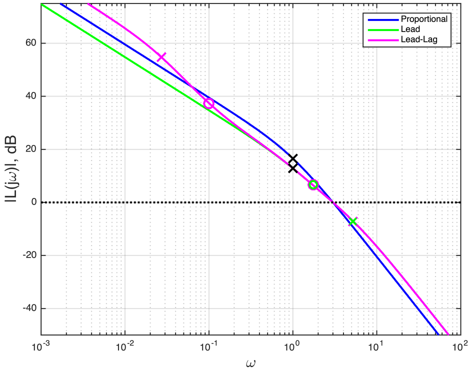
Example, Lead-Lag

- Type 1 plant of order 2: $G(s) = \frac{0.2}{s(s+1)}$
- Design goals:
  - Open loop crossover frequency at $\omega_c \approx 3$rads^{-1}.
  - Phase margin of 45°
  - Low-frequency disturbances attenuated by a factor of at least 40dB
- Our lead controller for this plant (green) achieves the first two goals
- The third goal corresponds to the requirement that $\lim_{s\to 0}\left|\frac{G(s)}{1+G_c(s)G(s)}\right| \leq 10^{-40/20} = 1/100$
- Since $G(s)$ is type-1, at low frequencies $G(s)$ is large and hence $\lim_{s\to 0}\left|\frac{G(s)}{1+G_c(s)G(s)}\right| \approx \lim_{s\to 0}\frac{1}{G_c(s)}$
- For our lead design, $\lim_{s\to 0}\frac{1}{G_c(s)} \approx \frac{5.2}{82.2\times 1.73} \approx \frac{1}{27.3}$
- Fails to meet specifications.
- Need to design a lag controller for $\tilde{G}(s) = G_{c,\text{lead}}(s)G(s)$ that increases the low frequency gain by $100/27.3 \approx 3.66$

Example, lead-lag

- Need $\alpha_{\text{lag}} = 3.66$.
- Place zero of lag compensator a factor of 30 below the desired crossover frequency; $z = 3/30 = 1/10$.
- Place pole of lag compensator at $p = z/\alpha \approx 0.027$
- Lead-lag compensator: $G_c(s) = 82.2\frac{s+0.1}{s+0.027}\frac{s+1.73}{s+5.2}$

Bode Mag Diagrams, open loop



Black x: frequency of plant pole;

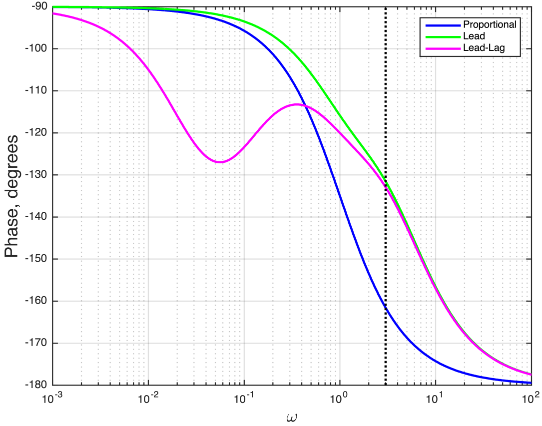
Green x and circle: frequencies of lead compensator pole and zero

Magenta x's and circles: freq's of lead-lag compensator poles and zeros

Same cross over frequency; lead and lead-lag have shallower slope

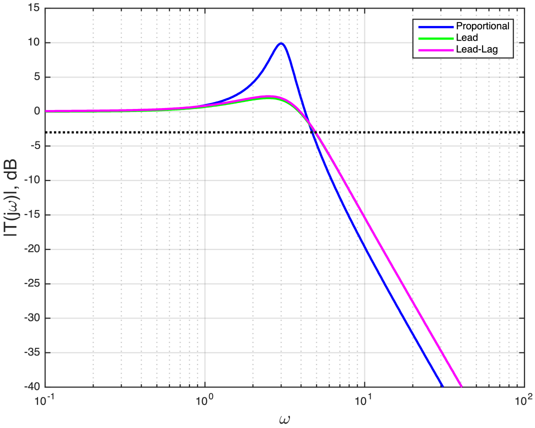
Lead-lag has larger low-frequency open-loop gain

Bode Phase Diagrams, open loop



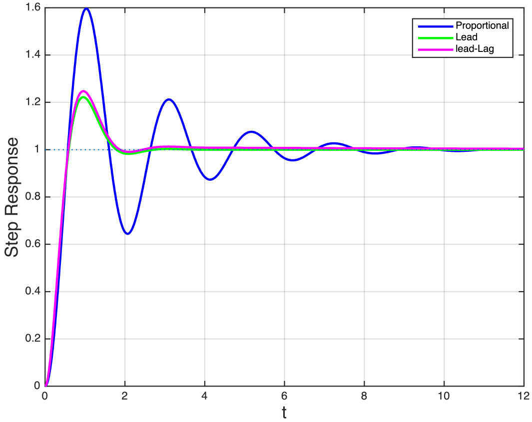
Observe additional phase from lead compensator and improved phase margin. By design, lead-lag does not reduce this much.

Bode Mag Diagrams, closed loop



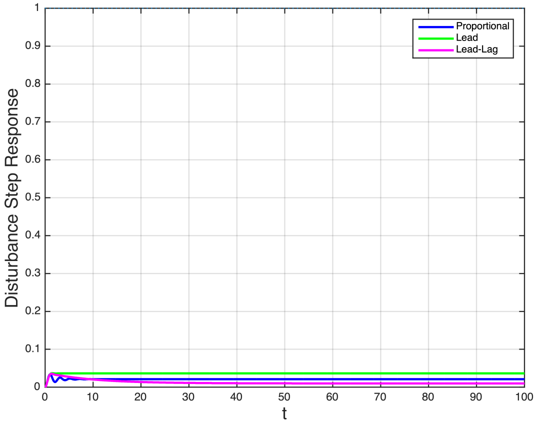
By design, lead-lag is similar to lead

Step Responses



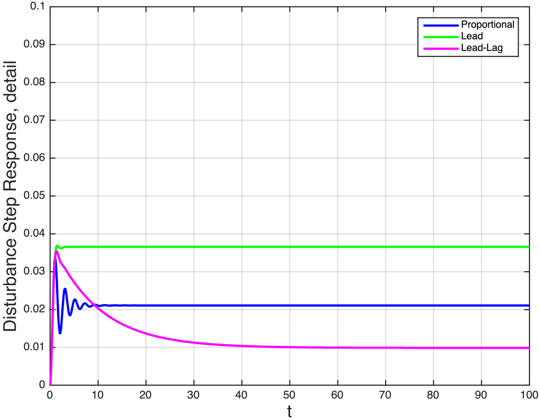
By design, lead-lag is similar to lead

Responses to step disturbance



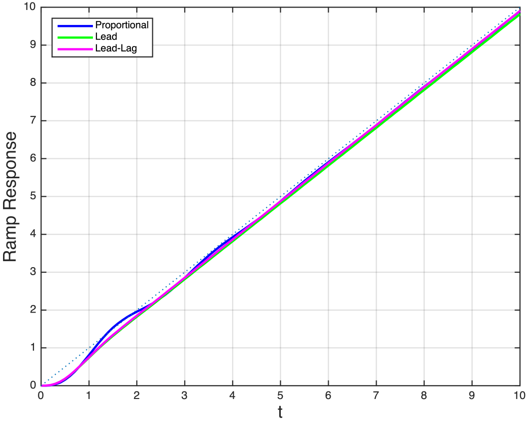
Lead-lag has better performance than lead due to larger low-frequency open-loop gain

Responses to step disturbance, detail

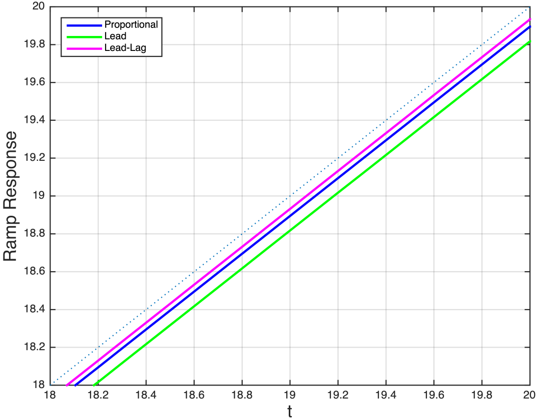


Lead-lag meets the requirement on mitigating low frequency disturbances

Ramp Reponse



Ramp Reponse, detail



$K_{v,\text{leadlag}} \approx 20.3 > K_{v,\text{prop}} \approx 9.5 > K_{v,\text{lead}} \approx 5.5$

Again, larger low-frequency open-loop gain plays the key role here.