FT representation of periodic signals

- Its sometimes quite ~~[illegible]~~ inconvenient that under the formal definition, periodic signals do not have a Fourier Transform.

- However, if we allow the FT to contain impulses we can get around this.

- Consider a periodic signal with ~~f~~ fundamental frequency ω_0.

$$x(t) = \sum_k X[k] e^{jk\omega_0 t}.$$

i.e, $x(t)$ is a linear combination of complex exponentials

- To find a FT of $x(t)$, we need to know the FT of $e^{jk\omega_0 t}$.

- We already have that $1 \xleftrightarrow{FT} 2\pi\delta(\omega)$
- Hence using the frequency shift property,

$$\underbrace{e^{jk\omega_0 t}}_{\text{complex exp in time}} \xleftrightarrow{FT} \underbrace{2\pi\delta(\omega - k\omega_0)}_{\text{spike in frequency.}}$$

[redacted]

So what is $X(j\omega)$?

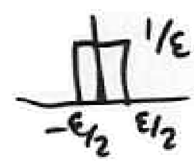
$$X(j\omega) = \int x(t)\, e^{j\omega t}\, dt$$

$$\sum_k X[k] \int e^{jk\omega_0 t} e^{-j\omega t}\, dt$$

$$= 2\pi \sum_k X[k]\ \delta(\omega - k\omega_0)$$

a weighted sum of spikes (Fig 4.4)

Recall that



[redacted]

$$\alpha\, \delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

and has area α

- We often adjust the [redacted] [redacted] size of the spikes to reflect the area.
- All spikes actually go off to ∞.

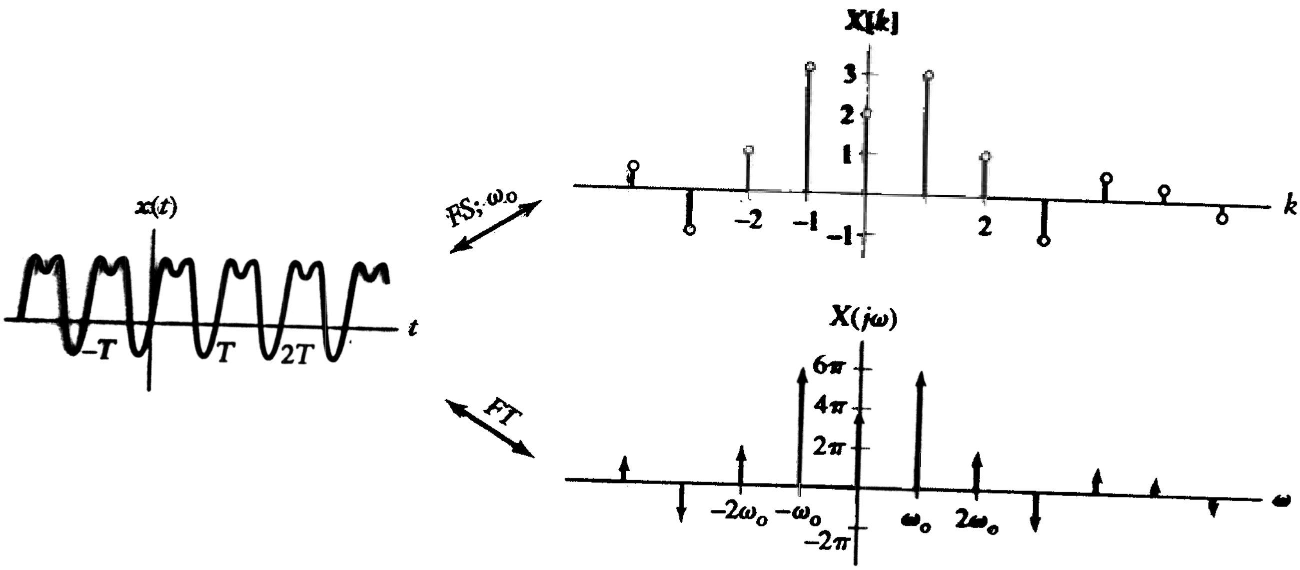


FIGURE 4.4 FS and FT representations for a periodic continuous-time signal.

[redacted]

What happens in discrete-time?

$$1 \xleftrightarrow{\text{DTFT}} 2\pi \sum_m \delta(\Omega - 2\pi m)$$

Note periodic in Ω with period 2π.

Therefore

$$e^{jk\Omega_0 n} \xleftrightarrow{\text{DTFT}} 2\pi \sum_m \delta(\Omega - k\Omega_0 - 2\pi m).$$

- Seeing as DTFT is periodic with period 2π, we usually focus on $-\pi < \Omega \leq \pi$.

- Today's discussion is most useful in the study of analog and digital communication systems in a continuous-time setting

- In this case, messages are multiplied by a periodic signal: (often called modulation of the periodic signal by the message)

- We know that for non-periodic signals,

  multiplication in ~~frequency~~ time $\xleftrightarrow{FT}$ convolution in frequency

  $$y(t) = g(t)x(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi} G(j\omega) * X(j\omega)$$

- What happens when $x(t)$ is periodic?

  Well! $X(j\omega) = 2\pi \sum_k X[k]\, \delta(\omega - k\omega_0)$

- Recall sifting property of impulse

  $$G(j\omega) * \delta(\omega - \omega_c) = \int G(j(\omega-\lambda))\, \delta(\lambda - \omega_c)\, d\lambda$$
  $$= G(j(\omega - \omega_c))$$

- Using this result + linearity, for periodic $x(t)$,

  $$y(t) = g(t)x(t) \xleftrightarrow{FT} Y(j\omega) = \sum_k X[k]\, G(j(\omega - k\omega_0$$

That is,

$$y(t) = g(t)x(t) \xleftrightarrow{FT} Y(j\omega) = \underbrace{\sum_k}_{\text{sum}} \underbrace{X[k]}_{\text{weighted}} \underbrace{G(j(\omega - k\omega_0))}_{\substack{\text{frequency shifted} \\ \text{FTs of } g(t)}}$$

EXAMPLE (Section 5.4). - TRANSMISSION SPECTRUM OF AN AM signal

- Consider a sinusoidal carrier

$$c(t) = A_c \cos\omega_c t$$

- and a message $m(t)$.
- In Amplitude modulation (AM) we transmit information by changing the amplitude of the carrier in proportion to the message

$$s(t) = A_c\left(1 + \bar{k}\,m(t)\right)\cos(\omega_c t)$$

- What is the spectrum of $s(t)$?

- $A_c \cos(\omega_c t)$ is periodic with fundamental frequency ~~ ~~, ω_c

Using standard formulae, (Euler)

$$c(t) = \frac{A_c}{2} e^{-j\omega_c t} + \frac{A_c}{2} e^{j\omega_c t}$$

- But this is already in the Form of a Fourier Series.

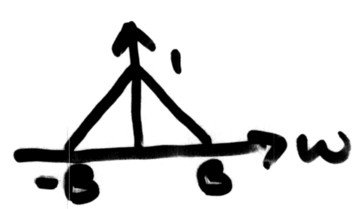
ie $$X[k] \begin{cases} \frac{A_c}{2} & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Now $s(t)$ $$\frac{A_c}{2} e^{-j\omega_c t} + \frac{A_c}{2} e^{j\omega_c t} + \frac{k A_c}{2} m(t) e^{-j\omega_c t} + \frac{k A_c}{2} m(t) e^{+j\omega_c t}$$

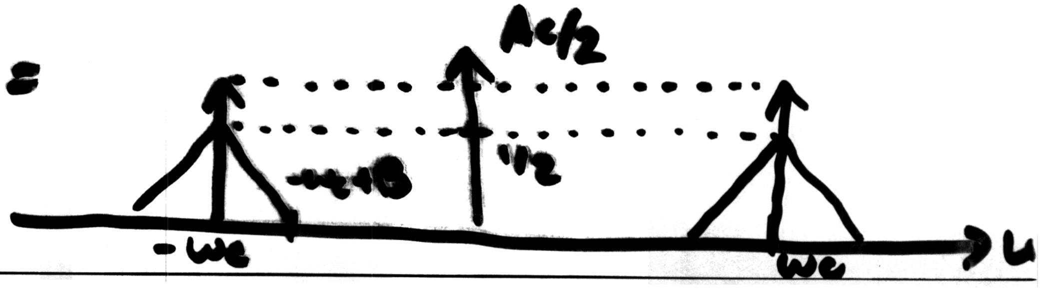
Hence.

$$S(j\omega) = \frac{A_c}{2} \delta(\omega + \omega_c) + \frac{A_c}{2} \delta(\omega - \omega_c) + \frac{k A_c}{2} M(j(\omega + \omega_c)) + \frac{k A_c}{2} M(j(\omega - \omega_c))$$

- If $M(j\omega) =$



Then $S(j\omega) =$



MORE OF THIS IN EE 3TR4.