Laplace Transforms

- We've already seen that some signals, eg, $e^{at}u(t)$, $a>0$ do not have a Fourier Transform. Also unstable systems do not have a frequency response. what to do?

  Modify the FT + develop the Laplace Transform.

  Recall that if $s = \sigma + j\omega$

$$e^{st} = e^{\sigma t}\cos(\omega t) + j e^{\sigma t}\sin(\omega t)$$

What happens if we apply e^{st} to an LT system?

$$y(t) = \int h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st}\int h(\tau)\, e^{-s\tau}\, d\tau$$

Define the transfer function $H(s) = \int h(\tau)\, e^{-s\tau}\, d\tau$

Then $y(t) = \underbrace{H(s)}_{\text{complex number indep. of } t}\ \underbrace{e^{st}}_{\text{complex exponential with same } \omega \text{ and same } \sigma}$

*Very much like the Fourier case, system only affects magnitude + phase

Now

$$H(s) = H(\sigma + j\omega) = \int h(t)\, e^{-st}\, dt$$

$$= \int h(t)\, e^{-\sigma t}\, e^{j\omega t}\, dt$$

Notes $\quad H(s)\big|_{\sigma=0} = H(j\omega) =$ frequency response

Define $\bar{h}(t) = e^{\sigma t} h(t)$

$$H(s)\big|_{s=\sigma+j\omega} = \bar{H}(j\omega)$$

This tells us how to do the inverse Laplace transform

we know $\quad \bar{h}\, t \xleftrightarrow{FT} H(j\omega) = H(s)$

$$\Rightarrow \quad e^{-\sigma t} h(t) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

$$h(t \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} H(s\; e^{st}\, ds$$

$s = \sigma + j\omega$

Note that ds $\quad$ s complex

Note also that we have to choose σ

ALL THE ABOVE WORKS N GENERAL

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \overset{L}{\longleftrightarrow} X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

So when does the Laplace Transform exist?
For what sorts of signals is $X(s)$ finite

- Does this also depend on s?

$$X(s) \quad \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$
$$= \int_{-\infty}^{\infty} \left(x(t)\, e^{-\sigma t}\right) e^{-j\omega t}\, dt$$

Fourier Transform of $x\,t\; e^{-\sigma t}$

$\Rightarrow$ Laplace Transform exists when

$$\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}\; dt < \infty$$

Hence for a given class of signals there will be a range of σ's such that the Laplace Transform exists This is called the Region of convergence (ROC)
The ROC depends only on σ, and hence it s a vertica strip in the $s = \sigma + j\omega$ plane