PROPERTIES OF BILATERAL LAPLACE TRANSFORM

- very similar to Fourier case, so we will just state them.
- However, we must track the ROC.

Linearity

If $x(t) \overset{L}{\longleftrightarrow} X(s)$ with RoC R_x

$y(t) \overset{L}{\longleftrightarrow} Y(s)$ with ROC R_Y,

$$\Rightarrow \quad ax(t) + by(t) \overset{L}{\longleftrightarrow} aX(s) + bY(s)$$

with ROC atleast $R_x \cap R_Y$

in a few cases ROC can be larger than $R_x \cap R_Y$, eg Ex 6.13

Time Shift

$$x(t-\tau) \overset{L}{\longleftrightarrow} e^{-s\tau} X(s)$$

ROC unchanged.

Differentiation in Time

$$\frac{d}{dt} x(t) \overset{L}{\longleftrightarrow} s X(s)$$

with ROC atleast R_x

Time Integration

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s}$$

with ROC $R_x \cap \mathrm{Re}\{s\} > c$

Note similarities to properties of Unilateral Laplace Transform. ROC replaces initial conditions

Scaling

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right),$$ ROC is set of s, such that $\frac{s}{a}$ is in R_x

S-domain shift

$$e^{s_0 t} x(t) \overset{L}{\longleftrightarrow} X(s-s_0)$$ ROC is set of s, such that $s-s_0$ is in R_x

Convolution

$$x(t) * h(t) \overset{L}{\longleftrightarrow} H(s)X(s),$$ Roc is at least $R_x \cap R_h$

Initial and Final Value Theorems

Initial: If $X(s) = \frac{N(s)}{D(s)}$ and order$(N(s)) <$ order$(D(s))$

then $$\lim_{s \to \infty} sX(s) = x(0^+)$$

Final: If $X(s)$ has all its poles in the left half plane, with at most one zero at $s=0$,

then $$\lim_{s \to 0} sX(s) = x(\infty)$$

INVERSION OF LAPLACE TRANSFORM

- Many practical signals and systems have Laplace. Transforms which are rational functions

ie $X(s) = \dfrac{N(s)}{D(s)}$.

$$x(t) = \frac{1}{2\pi j}\int x(t) e^{st}\, ds.$$

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$

- Furthermore. we know that.

$$X(s) = \frac{A}{s-p} \overset{L^{-1}}{\longleftrightarrow} x(t) = \begin{cases} A e^{pt} u(t), & \text{if } * \\ -Ae^{pt} u(-t), & \text{if } \square \end{cases}$$

$*$ ROC to right of p

$\square$ ROC to left of p

Thus if we can write.

$$X(s) = \sum_{k=1}^{N} \frac{A_k}{s-p_k}$$

Then we can invert each term individually and add them, using above relationship

Other terms which can occur in partial fraction expansion have inverses as follows:

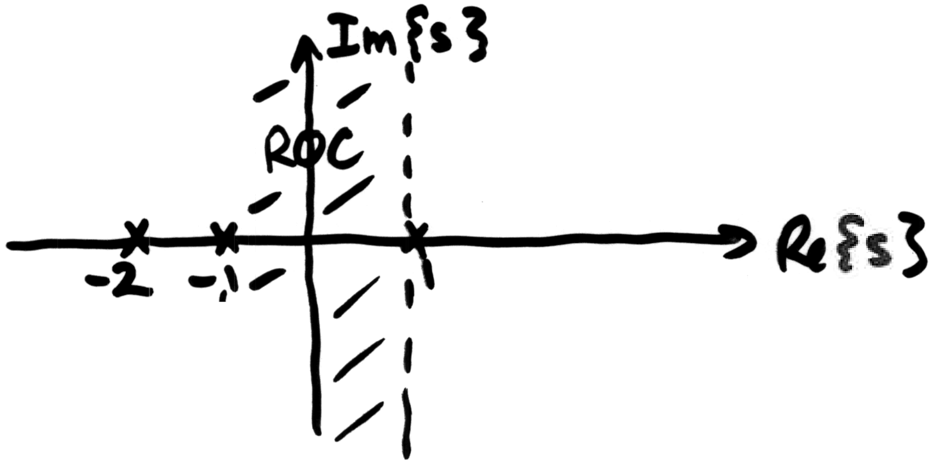
$$\frac{A}{(s-p_k)^n} \stackrel{L^{-1}}{\longleftrightarrow} \begin{cases} \frac{At^{n-1}}{(n-1)!} e^{p_k t} u(t), & \text{if ROC is to right of } p_k \\ \frac{-At^{n-1}}{(n-1)!} e^{p_k t} u(-t), & \text{if ROC is to left of } p_k \end{cases}$$

$$\frac{C_1(s-\alpha)}{(s-\alpha)^2 + \omega_0^2} \stackrel{L^{-1}}{\longleftrightarrow} \begin{cases} C_1 e^{\alpha t} \cos(\omega_0 t) u(t), & \text{if ROC to right of } \alpha \pm j\omega_0 \\ -C_1 e^{\alpha t} \cos(\omega_0 t) u(t), & \text{if ROC to left of } \alpha \pm j\omega_0 \end{cases}$$

EXAMPLE 6.15

Find $x(t)$ if $X(s) = \dfrac{-5s-7}{(s+1)(s-1)(s+2)}$,

with ROC $\quad < Re\{s\} <$



$$X(s) = \frac{A}{s+1} + \frac{A_2}{s-1} + \frac{A_3}{s+2}$$

$$\frac{1}{s+1} \quad \frac{2}{s-1} + \frac{1}{s+2}$$

First term has pole at $s=-$, ROC is to the right of this pole

$$\Rightarrow \frac{1}{s+1} \xleftrightarrow{L^{-1}} e^{-t}u(t)$$

Second term has pole at $s=1$, ROC is to left of this pole

$$\Rightarrow \frac{-2}{s-1} \xleftrightarrow{L^{-1}} 2e^{t}u(-t)$$

Third term has pole at $s=-2$ ROC is to the right

$$\Rightarrow \frac{1}{s+2} \xleftrightarrow{L^{-1}} e^{-2t}u(t)$$

$$\Rightarrow x(t) = e^{-t}u(t) + 2e^{t}u(-t) + e^{-2t}u(t)$$