PROPERTIES OF CONVOLUTION REPRESENTATION

- The impulse response completely characterizes the input-output behaviour of an LTI system
- We can calculate the output due to any input
- Hence the impulse response tells us a lot about system; eg, memory, causality, stability, etc.
- These properties + their proofs are very similar continuous-time + discrete-time systems.
- We will prove one and state the other.
- Try to prove the other for homework.

Memoryless LTI systems

- A system is memoryless if output depends only on current input

$$y[n] = \sum_k x[k]\,h[n-k]$$

Hint: Think of k as "input time"
n as "output time"

Set $m = n-k$

$$\Rightarrow \quad y[n] = \sum_m h[m]\,x[n-m]$$

The system is memoryless only if $y[n]$ depends on $x[n]$ only.

$\Rightarrow$ $h[m]$ must be zero for all $m \neq 0$

$\Rightarrow$ $h[k] = c\,\delta[k]$ if the system is memoryless

- Similarly in continuous time

$$y(t) = \int x(\tau)\,h(t-\tau)\,d\tau = \int x(t-\lambda)\,h(\lambda)\,d\lambda$$

System is memoryless only if $h(t) = c\,\delta(t)$

CAUSAL LTI SYSTEMS

Causal: output depends only on past and present inputs

$$y[n] = \sum_{m} h[m]\, x[n-m]$$

- Note that present input corresponds to $m=0$
  past inputs correspond to $m>0$
  future inputs correspond to $m<0$

- Hence for the system to be causal, we must have
  $$h[m] = 0, \quad m<0$$

- In that case, convolution sum can be simplified

$$y[n] = \sum_{m=0}^{\infty} h[m]\, x[n-m]$$

$$\sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

Causality in continuous time

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

$$\int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

$\Rightarrow$ system is causa if $h(t)=0, \quad t<0$

In that case, convolution integra simplifies to

$$y(t) = \int_{0}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

$$\int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

- Causal systems are non-anticipative

STABLE SYSTEMS

- A DT system is bounded-input bounded output (BIBO) stable if for all

$$|x[n]| \leq M_x < \infty$$

we have $|y[n]| \leq M_y < \infty$

- What property does the impulse response of an LTI stable system have?

- $$|y[n]| = \left| \sum_k h[k]\, x[n-k] \right|$$

- Recall, $|a+b| \leq |a| + |b|$

Hence $$|y[n]| \leq \sum_k |h[k]\, x[n-k]|$$

- Recall $|ab| \quad |a||b|$

Hence $$|y[n]| \leq \sum_k |h[k]|\,|x[n-k]|$$

If the input is bounded,

$$|x[n]| \leq M_x < \infty$$

then $|x[n-k]| \leq M_x$

and hence

$$|y[n]| \leq M_x \sum_k |h[k]|$$

Therefore, system is stable provided

$$\sum_k |h[k]| < \infty$$

Use similar steps to show that a continuous-time system is stable provided

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Example

- Is an LT system with $h[n] = u[n]$ BIBO stable?

- $u[n] = \begin{cases} & n \geq 0 \\ 0 & n < 0 \end{cases}$,

  so t looks ike a "sensible" function.

- However, $\sum_{k} h[k]| = \sum_{k=0}^{\infty} 1 \rightarrow \infty$

- Hence the system is not stable

  To see that that is true show that $x[n] = u[n]$ then $y[n]$ gets large as n gets large