Reconstruction of sampled signals

$$x(t) \longrightarrow \boxed{\text{Sampler}} \xrightarrow{x[n] = x(nT)} \boxed{\text{Reconstruction}} \longrightarrow \hat{x}(t).$$

Question what is the condition on T so that perfect reconstruction is possible?

Question 2 When it is possible, how do we do it?

Is it possible at all?

$x[n]$ tells us nothing about what happens between samples.
<u>Fig 4.27</u>

- If we only have the samples, how can we reconstruct $x(t)$? (We have already seen aliasing problem)

If ~~the sign~~ we know that the signal varies slowly between the samples (ie., the sampling is fast enough) we should be able to get pretty close by just joining the dots

How fast is fast enough?

What is the smartest way to join the dots.

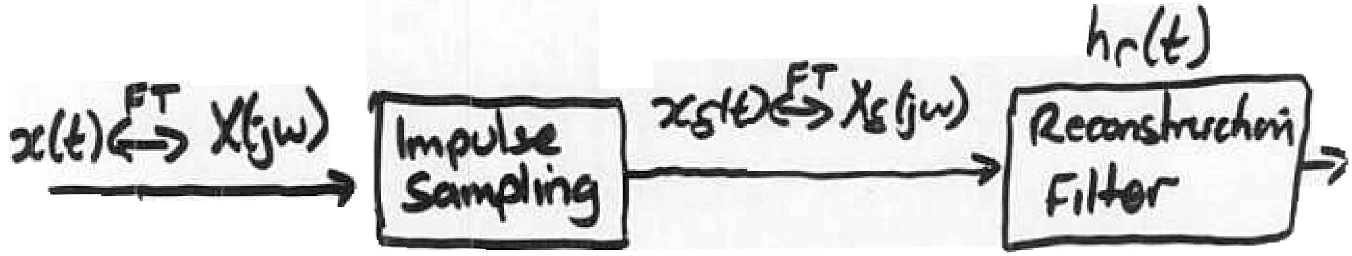
- Answers via Fourier representation

The key observation is that if

$X_s(j\omega)$ has the same shape as $X(j\omega)$

between $-\omega_s/2$ and $\omega_s/2$

Then we can reconstruct $x(t)$ perfectly just by filtering (Fig 4.29)



$$\hat{x}(t) \overset{FT}{\longleftrightarrow} \hat{X}(j\omega)$$

$$\hat{X}(j\omega) = H_r(j\omega)\, X_s(j\omega)$$

$$\hat{X}(j\omega) \quad X(j\omega) \text{ then } \hat{x}(t) = x(t)$$

Now state this formally

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) \quad \text{with} \quad X(j\omega) = 0, \ |\omega| > \omega_m$$

If $\omega_s > 2\omega_m$, *where* $\omega_s = 2\pi/T$ *then* $x(t)$ *is uniquely determined by* $x(nT)$.

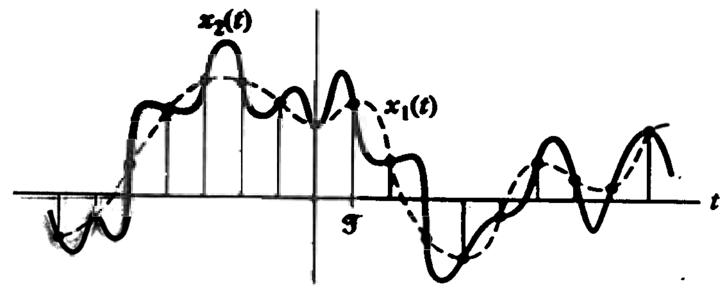


FIGURE 4.27 Two continuous-time signals, $x_1(t)$ (dashed line) and $x_2(t)$ (solid line), that h
the same set of samples.

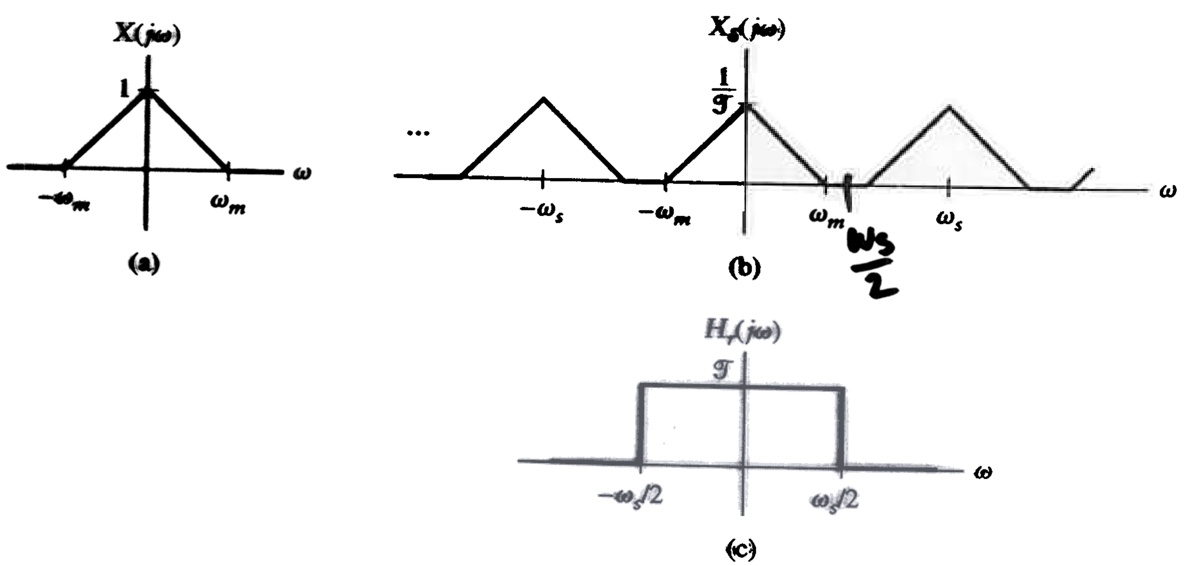


FIGURE 4.29 Ideal reconstruction. (a) Original signal spectrum. (b) Sampled signal spectrum.
(c) Frequency response of reconstruction filter.

The minimum sampling frequency, $2\omega_m$, is often called the Nyquist rate.

- This rate is usually expressed in Hertz, $\frac{2\omega_m}{2\pi}$

EXAMPLE

- What sampling rate should be chosen for CD audio?

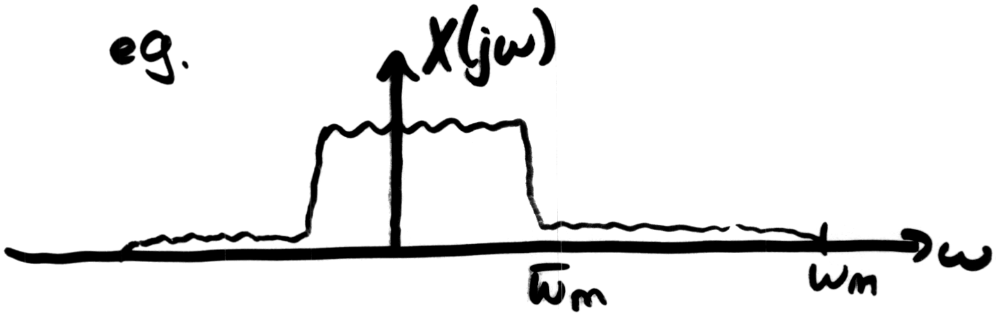
- Human ear can hear approx 20Hz – 20kHz.
- In a good design we should be able to perfectly represent all signals with frequency components up to 20kHz.
- $\Rightarrow$ minimum sampling rate is 40 k samples/sec
- Actual rate is 44.1 k samples/sec, to make reconstruction easier.

Choice of sampling rate

$\omega_s > 2\omega_m$.

What happens if ω_m is large, but we are only interested in the low frequency components?

eg.

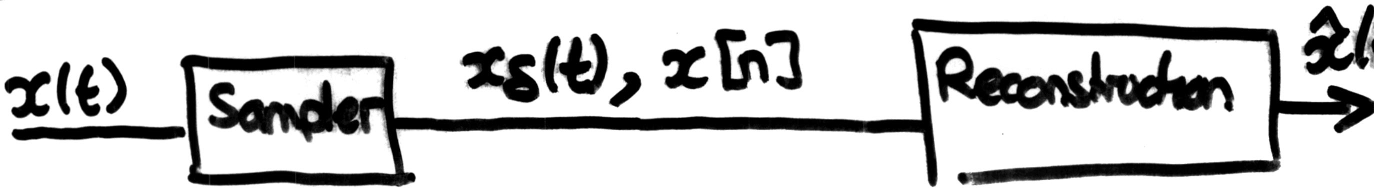


According to theory we must choose $\omega_s > 2\omega_m$, but if we are only interested in frequencies up to $\bar{\omega}_m$, we can filter the signal first

- Now sample $\tilde{x}(t)$ at $\omega_s > 2\bar{\omega}_m$

Anti alias filters appear in most ADCs

Ideal Reconstruction



We have seen

$$X_s(j\omega) = \frac{1}{T} \sum_k X(j\omega - k\omega_s)$$

If Nyquist criterion is satisfied, then we can simply "filter out" one "lobe" of $X_s(j\omega)$ to reconstruct $x(t$

In the ideal case, the filter has a frequency response

$$H_r(j\omega) = \begin{cases} T & |\omega| < \omega_s/2 \\ 0 & \omega| > \omega_s/2 \end{cases}$$

and $$\hat{X}(j\omega) = H_r(j\omega)\, X_s(j\omega)$$

In the time domain

$$\hat{x}(t) = h_r(t) * \left(\sum_k x[n]\, \delta(t - nT) \right)$$

$$= \int h_r(t-\tau) \sum_n x[n]\, \delta(\tau - nT)\, d\tau$$

$$\sum_n x[n]\, h_r(t - nT)$$

~~tx~~ Now $\quad h_r(t) = \dfrac{T \sin\left(\frac{\omega_s t}{2}\right)}{\pi t}$

$$= \operatorname{sinc}\left(\frac{\omega_s t}{2\pi}\right)$$

$$\Rightarrow \qquad \hat{x}(t) = \cancel{x} \sum_n x[n] \operatorname{sinc}\left(\frac{\omega_s}{2\pi}(t - nT)\right)$$

and $\quad \hat{x}(t) = x(t).$

This is called ideal bandlimited interpolation

Fig 4.30

- Can we build a filter with $h(t) = \operatorname{sinc}\left(\frac{\omega_s t}{2\pi}\right)$
- Well t is non-causal for a start
- In practice we build filters which approximate the frequency response of the sinc
- This approximation is easier and cheaper if is a bit bigger than $2\omega_m$.

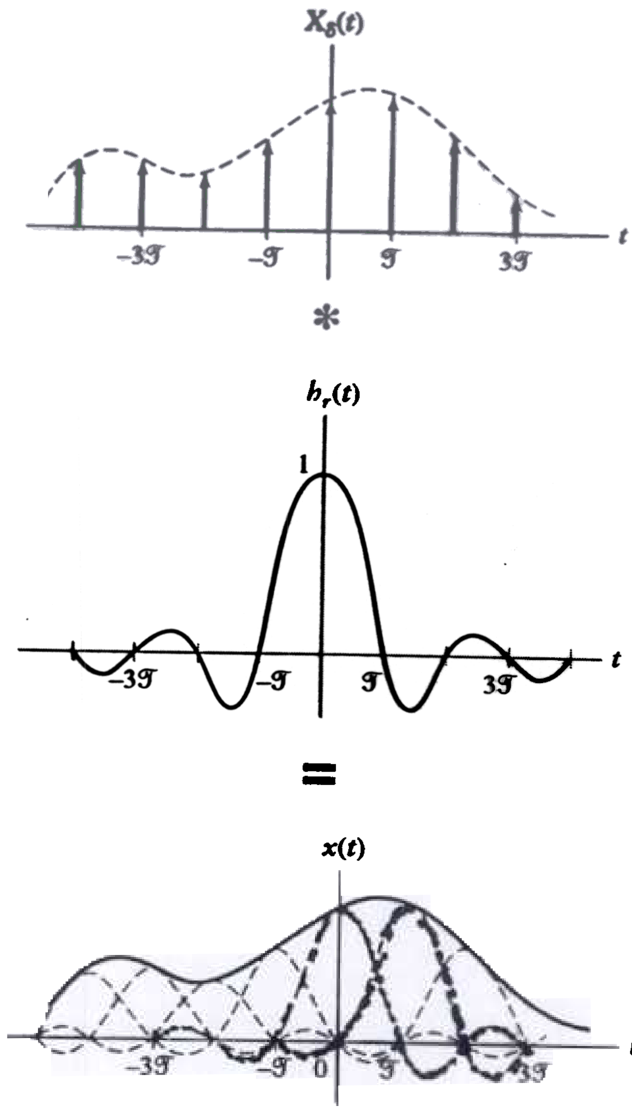


FIGURE 4.30 Ideal reconstruction in the time domain