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Generalized PI controllability

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Integral controllable systems can be stabilized by all sufficiently low-gain integral controllers. The concept of integral controllability is extended to two generalized stability regions and to proportional controllability and PI controllability with respect to those regions. Furthermore, closed eigenvalue formulae are derived for the maximal low integral gain and the maximal low proportional gain for generalized stability. Related eigenvalue formulae are then used in combination to find all low-gain PI controllers that place the closed-loop eigenvalues in the desired region.

1. Introduction

A stable system is said to be *integral controllable* if it can be stabilized by all sufficiently low gain integral controllers. Conditions for a system to be integral controllable were given by Morari (1985) and exact formulae for the maximal low gain for stability were derived under mild conditions by Mustafa (1994*a*, 1994*c*). Integral controllability of a system is a useful property in the design of simple decentralized controllers (see Grosdidier *et al.* 1985, Morari and Zafiriou 1989, Campo and Morari 1994). By correcting Lemma 10·4 of Lunze (1989) it can be shown that a stable system can be stabilized by all sufficiently low-gain PI controllers (i.e. is *PI controllable*) if and only if it is integral controllable. Results on the maximal range of stabilizing PI gains have been presented by Mustafa (1994*b*).

It is well known (see for example, Šiljak 1969, Gutman 1990 and Ackermann *et al.* 1993) that by using a generalized notion of stability it is possible to enforce certain useful performance criteria. For example, a minimum damping factor constraint can be satisfied by keeping all the closed-loop eigenvalues in a left sector of the complex plane. Similarly, a maximum oscillation frequency (damped natural frequency) constraint can be satisfied by keeping all the closed-loop eigenvalues in a horizontal band in the complex plane. In the present paper the integral controllability condition of Morari (1985) and the maximal low integral gain formulae of Mustafa (1994*a*) are extended to cover such generalized stability constraints, based on the preliminary work in Mustafa and Davidson (1994). Furthermore, we derive a formula for the maximal low proportional gain for generalized stability, and provide a method for finding the maximal low gain region in PI gain space for which the closed-loop eigenvalues are in the desired region of the complex plane.

The paper is organized as follows. In §2 we introduce the control system of interest and give a brief summary of the relevant mathematical tools. In §3 we give conditions for integral controllability with respect to a left sector. We then derive a closed formula for the maximal low integral gain with respect to the left sector by using block Kronecker algebra (Hyland and Collins 1989) to modify a guardian map (Saydy *et al.*

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1990). This formula generalizes the formula of Mustafa (1994*a*, 1994*c*). Using similar techniques we also derive closed formulae for the maximal low integral gain with respect to the horizontal band and for the maximal low integral gain with respect to the intersection of the left sector and the horizontal band. In §4 we use a guardian map approach to calculate the maximal low proportional gains for the left sector and the horizontal band, and in §5 we show how to construct the maximal low gain region in PI gain space for the sector and the band.

2. Preliminaries

In the present paper we consider the negative unity feedback connection of the controller $k(s)\,I_m$ to an $m\times m$ system $G(s) = D + C(sI-A)^{-1}B$, as shown in Fig. 1. Of course, $G(s)$ can be a pre- (or a post-) compensated plant, as in Morari (1985). In §3, $k(s)$ is the integral controller $k(s) = k_{\mathrm{I}}/s$; in §4, $k(s)$ is the proportional controller $k(s) = k_{\mathrm{P}}$; and in §5, $k(s)$ is the PI controller $k(s) = k_{\mathrm{P}} + k_{\mathrm{I}}/s$.

Given an open region Γ in the complex plane, we say that a (square) matrix is *in* Γ if all its eigenvalues are in Γ. Similarly, given a system $G(s) = D + C(sI-A)^{-1}B$, we say that $G(s)$ is *in* Γ if A is in Γ. If $\bar{D} + \bar{C}(sI-\bar{A})^{-1}\bar{B}$ is the closed-loop transfer function from r to y in Fig. 1, we call the eigenvalues of $\bar{A}$ the *closed-loop eigenvalues*. We say that the controller Γ*-assigns* $G(s)$ if the closed-loop dynamics matrix $\bar{A}$ is in Γ. Of course, if Γ is the (open) left half-plane then Γ-assignment is equivalent to internal stability.

Let $\otimes$ and $\oplus$ denote the usual Kronecker product and Kronecker sum respectively (see Brewer 1978, for a survey), and let $\bar{\otimes}$ and $\bar{\oplus}$ denote the block Kronecker product yand block Kronecker sum respectively as defined by Hyland and Collins (1989). In the present paper we are interested in 2×2 block matrices. In this case, for $n\times n$ matrices A and B partitioned identically as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

the block Kronecker product $A \,\bar{\otimes}\, B$ is the $n^2 \times n^2$ matrix

$$A \,\bar{\otimes}\, B := \begin{bmatrix} A_{11}\otimes B_{11} & A_{11}\otimes B_{12} & A_{12}\otimes B_{11} & A_{12}\otimes B_{12} \\ A_{11}\otimes B_{21} & A_{11}\otimes B_{22} & A_{12}\otimes B_{21} & A_{12}\otimes B_{22} \\ A_{21}\otimes B_{11} & A_{21}\otimes B_{12} & A_{22}\otimes B_{11} & A_{22}\otimes B_{12} \\ A_{21}\otimes B_{21} & A_{21}\otimes B_{22} & A_{22}\otimes B_{21} & A_{22}\otimes B_{22} \end{bmatrix}$$

and the block Kronecker sum $A \,\bar{\oplus}\, B$ is the $n^2\times n^2$ matrix $A \,\bar{\oplus}\, B := A \,\bar{\otimes}\, I_n + I_n \,\bar{\otimes}\, B$ (where, of course, I_n is partitioned conformally with A and B). Whilst the basic properties of Kronecker and block Kronecker algebra (as in Brewer 1978, and Hyland and Collins 1989, respectively) will be used freely, we draw attention to the fact that the eigenvalues of $A \otimes B$ are the n^2 pairwise products of the eigenvalues of A and B and the eigenvalues of both $A \oplus B$ and $A \,\bar{\oplus}\, B$ are the n^2 pairwise sums of the eigenvalues of A and B.

For a matrix M, let M^* denote its conjugate transpose. If M is square, let $\lambda_{\mathrm{real}}(M)$ denote any real eigenvalue of M and let $\lambda^{+}_{\max}(M)$ denote the largest positive real

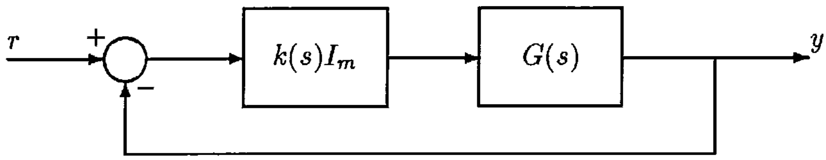

Figure 1. Closed-loop system.

eigenvalues of M, or 0^+ if M has no positive real eigenvalues. If M and N are square matrices of the same dimension, let $\lambda_{\mathrm{real}}(M, N)$ denote any finite real generalized eigenvalue of M (i.e. any finite real solution to det $(M - \lambda N) = 0$).

3. Generalized integral controllability

In this section we consider the negative unity feedback connection of the integral controller $k_I I_m/s$ to $G(s) = D + C(sI - A)^{-1}B$. The closed-loop dynamics matrix is

$$\bar{A}(k_I) := \begin{bmatrix} A & k_I B \\ -C & -k_I D \end{bmatrix} \tag{1}$$

The definition of integral controllability of Morari (1985) is generalized to a region Γ in the following way.

Definition 3.1: An $m \times m$ system $G(s)$ in Γ is said to be *Γ-integral controllable* if there exists a $\hat{k}_I > 0$ such that the integral controller $k_I I_m/s$ Γ-assigns $G(s)$ for all $k_I \in (0, \hat{k}_I)$. □

We will use the term *radius of Γ-integral controllability* to mean the largest value of $\hat{k}_I$ such that $k_I I_m/s$ Γ-assigns $G(s)$ for all $k_I \in (0, \hat{k}_I)$, and will denote it by $\hat{k}_I^{\max}$. The region Γ to which $\hat{k}_I^{\max}$ refers will be clear from the context.

In the following subsections we give closed formulae for $\hat{k}_I^{\max}$ for the cases where Γ is a left sector, a horizontal band an intersection of the sector and the band. We also provide some additional results required for §5.

3.1. *Integral controllability in a left sector*

In this section we study integral control with respect to the open left sector. The left sector is defined by

$$\Gamma = \{z : z \in \mathbb{C},\ \pi/2 + \alpha < \arg(z) < 3\pi/2 - \alpha\}$$

where $0 \leqslant \alpha < \pi/2$, and is illustrated in Fig. 2.

The following proposition gives conditions for left-sector integral controllability.

Proposition 3.1: *An $m \times m$ system $G(s)$ in the left sector is left-sector integral controllable if $-G(0)$ is in the left sector, and only if $-G(0)$ is in the closed left sector less the origin. If $m = 1$ then $G(s)$ is left-sector integral controllable if and only if $G(0) > 0$.*

Proof: For the proof see Appendix A.

Remark 3.1: For $\alpha = 0$, the above result collapses to the standard result on (left half-

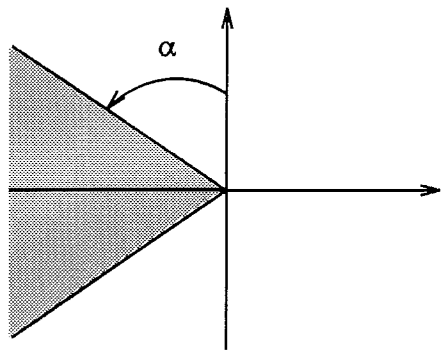

Figure 2. The left sector.

plane) integral controllability. That is, $G(s)$ is integral controllable if $-G(0)$ is in the left half-plane, and only if $-G(0)$ is in the closed left half-plane less the origin (Morari 1985). □

Remark 3.2: In a similar way to the standard results on (left half-plane) integral controllability (Morari 1985), Proposition 3.1 gives no information about the left-sector integral controllability of systems for which $-G(0)$ is in the closed left sector and is non-singular but has eigenvalues on the sector boundary. To illustrate that case, consider the systems

$$G_1(s) = \begin{bmatrix} 0 & -\dfrac{s+1}{s+10} \\ \dfrac{s+1}{s+10} & -2\cos\phi\dfrac{s+1}{s+10} \end{bmatrix} \quad \text{and} \quad G_2(s) = \begin{bmatrix} 0 & -\dfrac{s-1}{s+10} \\ \dfrac{s-1}{s+10} & 2\cos\phi\dfrac{s-1}{s+10} \end{bmatrix}$$

where $\phi := \pi/2+\alpha$. The eigenvalues of both $-G_1(0)$ and $-G_2(0)$ are $0{\cdot}1\,\mathrm{e}^{\pm j\phi}$ and hence are on the boundary of the left sector (defined by α). However, the closed-loop dynamics for $k_I I_2/s$ with $G_1(s)$ is in the left sector for all $k_I > 0$ and the closed-loop dynamics matrix for $k_I I_2/s$ with $G_2(s)$ is not in the left sector for any $k_I > 0$. Details of the corresponding root locus diagrams for the case of $\alpha = \pi/6$ are given in Fig. 3. For the case where $\alpha = 0$ so that $\phi = \pi/2$, the systems $G_1(s)$ and $G_2(s)$ are equal to the systems $H_2(s)$ and $H_3(s)$ in Example 4 of Grosdidier *et al.* (1985) respectively. That example also shows that the necessary and sufficient condition for (left half-plane) integral controllability of Lunze (1985) is only sufficient for $m > 1$ and hence that the result of Lunze (1985) is equivalent to that of Morari (1985). □

Let λ_i ($1 \leqslant i \leqslant n+m$) be the eigenvalues of $\bar{A}(k_I)$ in (1). A *critical integral gain* (*for the left sector*) is defined to be any finite real value of k_I such that

$$\mathrm{e}^{j\alpha}\lambda_p + \mathrm{e}^{-j\alpha}\lambda_q = 0$$

for some p, q ($1 \leqslant p,\ q \leqslant n+m$). When k_I is critical, $\bar{A}(k_I)$ has at least one pair of eigenvalues with the same magnitude that subtend an angle of $\pi-2\alpha$ at the origin or $\bar{A}(k_I)$ has at least one eigenvalue at the origin.

In part (*a*) of the following proposition, we give a formula for the radius of left-sector integral controllability. In parts (*b*) and (*c*) we give formulae for all the critical integral gains, for later use. The proof is in Appendix B. The basic idea is to show that the critical gains are the finite gains for which $\nu(k_I) = 0$, where $\nu(k_I)$ is a suitably constructed block-structured guardian map. The natural form of $\nu(k_I) = 0$ is that of a

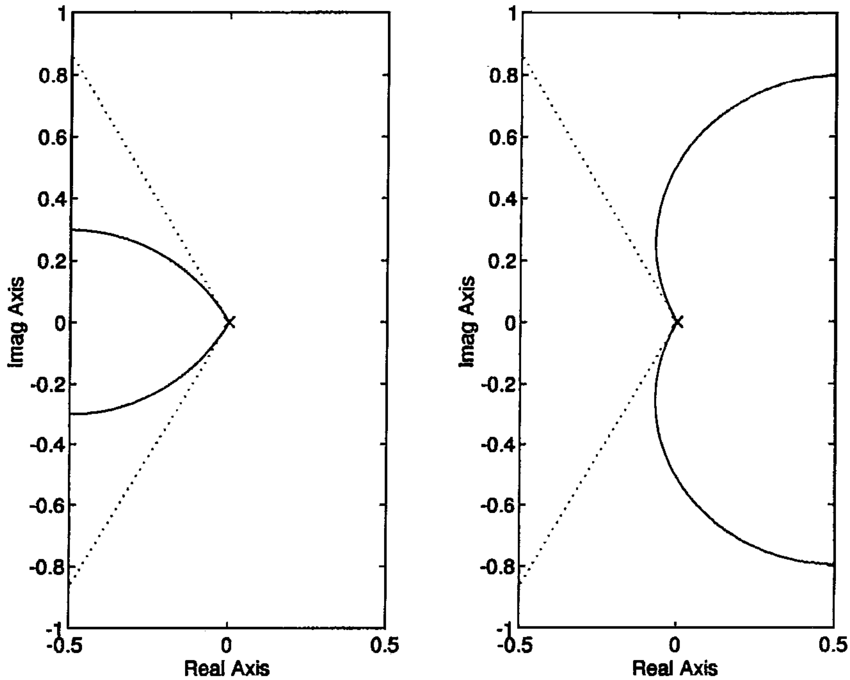

Figure 3. Details of the integral gain root loci for $G_1(s)$ (left) and $G_2(s)$ (right) in Remark 3.2.

generalized eigenvalue problem, but under certain mild assumptions Schur-type formulae for the determinant of a 2×2 block matrix can be used to reformulate $v(k_I) = 0$ as an eigenvalue problem of reduced dimension.

Proposition 3.2: *Let $G(s) = D + C(sI - A)^{-1} B$ be an n-state $m \times m$ system.*

(*a*) *If $G(s)$ is in the left sector and $-G(0)$ is in the left sector, then the radius of left-sector integral controllability is given by*

$$\hat{k}_I^{\max} = \frac{1}{\lambda_{\max}^{+}(S(\alpha))}$$

where $S(\alpha)$ is the $2mn \times 2mn$ matrix

$$S(\alpha) := \begin{bmatrix} e^{-j\alpha} A^{-1} \otimes e^{-j\alpha} D & 0 \\ 0 & e^{j\alpha} D \otimes e^{j\alpha} A^{-1} \end{bmatrix}$$

$$+ \begin{bmatrix} A^{-1}B \otimes I_m & e^{-j\alpha} A^{-1} \otimes e^{-j\alpha} C \\ I_m \otimes A^{-1}B & e^{j\alpha} C \otimes e^{j\alpha} A^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} (e^{j\alpha} G(0) \oplus e^{-j\alpha} G(0))^{-1} & 0 \\ 0 & -(e^{j\alpha} A \oplus e^{-j\alpha} A)^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} CA^{-1} \otimes e^{-j\alpha} G(0) & e^{j\alpha} G(0) \otimes CA^{-1} \\ I_n \otimes e^{-j\alpha} B & e^{j\alpha} B \otimes I_n \end{bmatrix}$$

(*b*) *All the critical gains are given by*

$$\lambda_{\text{real}}\left(e^{j\alpha}\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \bar{\oplus}\, e^{-j\alpha}\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix},\ e^{j\alpha}\begin{bmatrix} 0 & -B \\ 0 & D \end{bmatrix} \bar{\oplus}\, e^{-j\alpha}\begin{bmatrix} 0 & -B \\ 0 & D \end{bmatrix}\right)$$

(*c*) *If both* $e^{j\alpha}A \oplus e^{-j\alpha}A$ *and* $e^{j\alpha}G(0) \oplus e^{-j\alpha}G(0)$ *are non-singular, than all the critical integral gains are given by the finite values of*

$$\frac{1}{\lambda_{\text{real}}(S(\alpha))}$$

together with 0.

Proof: For the proof see Appendix B. □

Note that the mild assumptions in Proposition 3.2(*c*) allow a significant reduction in the dimension of the eigenvalue formulae for the critical integral gains. Indeed, $S(\alpha)$ is of dimension $2mn \times 2mn$ whereas the dimension of the generalized eigenvalue problem in Proposition 3.2(*b*) is $(n+m)^2 \times (n+m)^2$.

Remark 3.3. For $\alpha = 0$, Proposition 3.2(*a*) solves the same problem as Mustafa (1994*a*, 1994*c*) (i.e. it gives the radius of (left half-plane) integral controllability). However, Proposition 3.2(*a*) does not require the additional technical assumptions needed by Mustafa (1994*a*, 1994*c*). For example, Proposition 3.2(*a*) is valid for strictly proper systems in contrast to the results of Mustafa (1994*a*, 1994*c*). When $G(s)$ is strictly proper $S(0)$ simplifies to

$$S(0) = \begin{bmatrix} A^{-1}B \otimes I_m & A^{-1} \otimes C \\ I_m \otimes A^{-1}B & C \otimes A^{-1} \end{bmatrix}$$
$$\times \begin{bmatrix} (G(0) \oplus G(0))^{-1} & 0 \\ 0 & -(A \oplus A)^{-1} \end{bmatrix}\begin{bmatrix} CA^{-1} \otimes G(0) & G(0) \otimes CA^{-1} \\ I_n \otimes B & B \otimes I_n \end{bmatrix} \quad \square$$

The application of Proposition 3.2 is now illustrated by two examples. In the first example parts (*a*) and (*c*) of Proposition 3.2 are analytically verified for a simple system by using root locus roles to calculate the critical integral gains and show that these are the reciprocals of the real eigenvalues of $S(\alpha)$. In the second example the proposition is used to solve numerically an illustrative multivariable problem.

Example 3.1: Consider the left sector with $\pi/4 \leqslant \alpha \leqslant \pi/2$ and the system with $A = -1$, $B = 1$, $C = 1$ and $D = 1$ so that

$$G(s) = \frac{s+2}{s+1}$$

This system is simple enough for the integral gain root loci to be calculated exactly, as shown in Fig. 4. The integral gain root loci consist of a circle of radius $\sqrt{2}$, centred on -2, together with the indicated parts of the real axis.

Zero is obviously a critical integral gain as there is a closed-loop eigenvalue at the origin when $k_I = 0$. The non-zero critical integral gains are the gains at the points P and Q in Fig. 4. It is a simple exercise in elementary geometry to express, in terms of α, the distances from P and Q to the open-loop poles and zero. Applying the standard calibration rule for root locus, it follows that the critical integral gains are

$$0,\ 1 - 2\cos 2\alpha \mp 2\sin\alpha(-2\cos 2\alpha)^{1/2} \qquad (2)$$

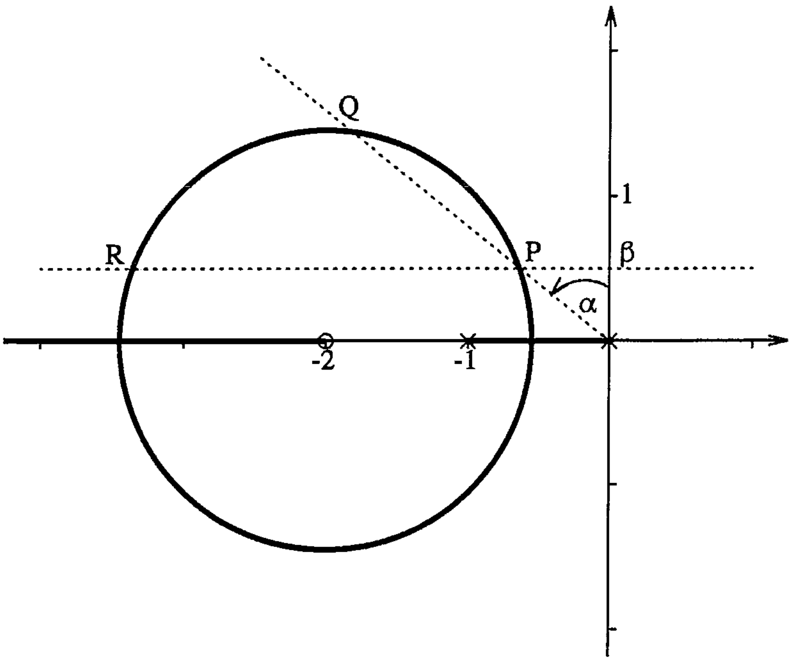

Figure 4. Integral gain root loci for Example 3.1.

Alternatively, using the formula in Proposition 3.2, we find that

$$S(\alpha)=\begin{bmatrix} -e^{-j2\alpha}(1-j\tan\alpha) & j\tan\alpha \\ -j\tan\alpha & -e^{j2\alpha}(1+j\tan\alpha) \end{bmatrix}$$

and hence

$$\lambda_{\text{real}}(S(\alpha)) = 1-2\cos 2\alpha \pm 2\sin\alpha(-2\cos 2\alpha)^{1/2} \tag{3}$$

Inverting (3) and rationalizing the denominator gives the non-zero term in (2) and hence parts (*a*) and (*c*) of Proposition 3.2 are verified for this simple system. □

Example 3.2: Consider the five-state 3 × 3 system with state-space matrices

$$A=\begin{bmatrix} -35{\cdot}27 & 7{\cdot}09 & 1{\cdot}51 & -5{\cdot}47 & 13{\cdot}44 \\ -72{\cdot}42 & 14{\cdot}36 & 4{\cdot}28 & -12{\cdot}16 & 27{\cdot}98 \\ -1{\cdot}78 & 1{\cdot}53 & 0{\cdot}20 & -1{\cdot}33 & 0{\cdot}31 \\ -60{\cdot}06 & 10{\cdot}52 & 3{\cdot}36 & -9{\cdot}03 & 24{\cdot}85 \\ -58{\cdot}10 & 12{\cdot}92 & 2{\cdot}92 & -10{\cdot}10 & 21{\cdot}23 \end{bmatrix},$$

$$B=\begin{bmatrix} -0{\cdot}75 & 0{\cdot}40 & -0{\cdot}40 \\ -0{\cdot}77 & -0{\cdot}08 & 0{\cdot}05 \\ -0{\cdot}82 & 0{\cdot}49 & -0{\cdot}07 \\ -0{\cdot}49 & 0{\cdot}34 & -0{\cdot}05 \\ -1.02 & 0 & -0{\cdot}47 \end{bmatrix},$$

$$C=\begin{bmatrix} 0{\cdot}58 & -1{\cdot}56 & -0{\cdot}27 & 0{\cdot}22 & 0{\cdot}31 \\ 1{\cdot}79 & 0{\cdot}25 & 0{\cdot}89 & -1{\cdot}02 & -1{\cdot}12 \\ 0{\cdot}97 & 0{\cdot}56 & -0{\cdot}06 & -0{\cdot}01 & -1{\cdot}12 \end{bmatrix}, \quad D=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

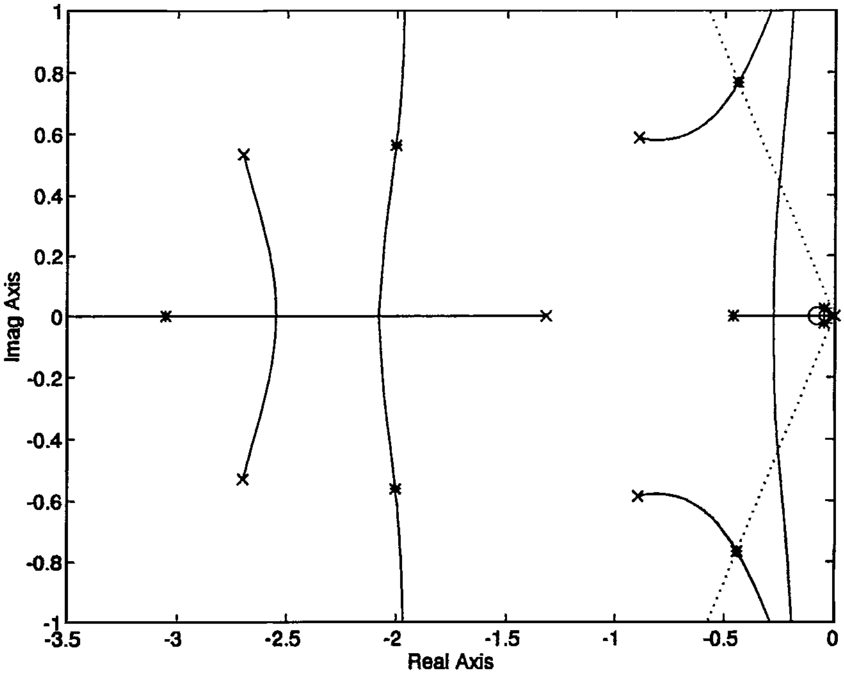

Figure 5. Integral gain root loci for Example 3.2, where × and * denote the closed-loop eigenvalues for $k_{\rm I} = 0$ and $k_{\rm I} = \hat{k}_{\rm I}^{\max}$ respectively.

Suppose that we wish to see how far we can increase integral action from zero whilst keeping the damping factor greater than $1/2 = \sin\pi/6$. In other words, find $\hat{k}_{\rm I}^{\max}$ for the 30° left sector. Using Proposition 3.2 (*a*) we find that $\hat{k}_{\rm I}^{\max} = 0{\cdot}1623$. To confirm this calculation, the integral gain root loci of the system are plotted in Fig. 5. It can be seen that the closed-loop eigenvalues do indeed satisfy the damping constraint for all $k_{\rm I} \in (0, \hat{k}_{\rm I}^{\max})$, and that the first violation of the damping constraint occurs when $k_{\rm I} = \hat{k}_{\rm I}^{\max}$. □

3.2. *Integral controllability in a horizontal band*

We now present a complementary analysis to that in the previous subsection, by considering integral control with respect to the open horizontal band in the complex plane defined by

$$\Gamma = \{z : z \in \mathbb{C},\ -\beta < \mathrm{Imag}(z) < \beta\}$$

where $\beta > 0$. This region is illustrated in Fig. 6.

By continuity of the eigenvalues of $\bar{A}(k_{\rm I})$ it is immediate that all systems in the horizontal band are horizontal-band integral controllable. Let λ_i $(1 \leqslant i \leqslant n+m)$ be the eigenvalues of $\bar{A}(k_{\rm I})$ in (1). A *critical integral gain* (*for the horizontal band*) is defined to be any finite real value of $k_{\rm I}$ such that

$$\lambda_p - \lambda_q = \mathrm{j}2\beta$$

for some p, $q (1 \leqslant p,\ q \leqslant n+m)$. When $k_{\rm I}$ is critical $\bar{A}(k_{\rm I})$ has at least one pair of eigenvalues with the same real part and imaginary parts, which differ by 2β.

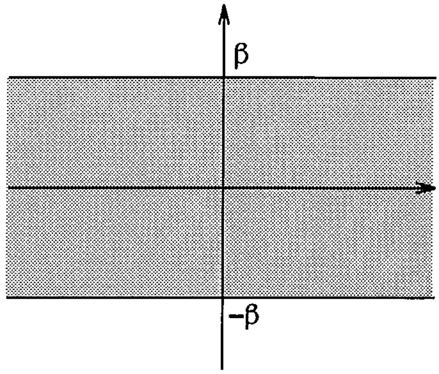

Figure 6. The horizontal band.

In part (*a*) of the following proposition we give a formula for the radius of horizontal-band integral controllability. In parts (*b*) and (*c*) we give formulae for all the critical integral gains, for use in § 5.

Proposition 3.3: *Let $G(s) = D + C(sI - A)^{-1}B$ be an n-state $m \times m$ system.*

(*a*) *If $G(s)$ is in the horizontal band, then the radius of horizontal-band integral controllability is given by*

$$k_{\mathrm{I}}^{\max} = \frac{1}{\lambda_{\max}^{+}(H(\beta))}$$

where $H(\beta)$ is the $(2mn + m^2) \times (2mn + m^2)$ matrix

$$H(\beta) := \begin{bmatrix} -A_{\mathrm{l}}^{-1} \otimes D & 0 \\ 0 & D \otimes A_{\mathrm{r}}^{-1} \\ (CA_{\mathrm{l}}^{-1} \otimes -G(j2\beta))/(\mathrm{j}2\beta) & -(G(-\mathrm{j}2\beta) \otimes CA_{\mathrm{r}}^{-1})/(\mathrm{j}2\beta) \end{bmatrix}$$
$$\times \begin{bmatrix} -A_{\mathrm{l}}^{-1}B \otimes I_m \\ I_m \otimes A_{\mathrm{r}}^{-1}B \\ (G(-\mathrm{j}2\beta) \oplus -G(j2\beta))/(\mathrm{j}2\beta) \end{bmatrix} + \begin{bmatrix} A_{\mathrm{l}}^{-1} \otimes C \\ -C \otimes A_{\mathrm{r}}^{-1} \\ CA_{\mathrm{l}}^{-1} \otimes CA_{\mathrm{r}}^{-1} \end{bmatrix}$$
$$\times ((A + \mathrm{j}\beta I_n) \oplus (-A + \mathrm{j}\beta I_n))^{-1} [-I_n \otimes B \quad B \otimes I_n \quad 0]$$

and $A_{\mathrm{l}} := A + \mathrm{j}2\beta I_n$ and $A_{\mathrm{r}} := -A + \mathrm{j}2\beta I_n$.

(*b*) *All the critical integral gains are given by*

$$\lambda_{\mathrm{real}}\left(\begin{bmatrix} A + \mathrm{j}\beta I_n & 0 \\ -C & \mathrm{j}\beta I_m \end{bmatrix} \bar{\oplus} \begin{bmatrix} -A + \mathrm{j}\beta I_n & 0 \\ C & \mathrm{j}\beta I_m \end{bmatrix}, \begin{bmatrix} 0 & -B \\ 0 & D \end{bmatrix} \bar{\oplus} \begin{bmatrix} 0 & B \\ 0 & -D \end{bmatrix}\right)$$

(*c*) *If $(A + j\beta I_n) \oplus (-A + j\beta I_n)$, A_l and A_r are non-singular, then all the critical integral gains are given by the finite values of*

$$\frac{1}{\lambda_{\mathrm{real}}(H(\beta))}$$

Proof: For the proof see Appendix C. □

Although the formulae in Proposition 3.3 are valid when $D = 0$, more compact formulae can be derived in that case. The proof is similar to that for Proposition 3.3 with an extra application of Schur's formula for the determinant of a 2×2 block matrix.

Corollary 3.4: *If $D = 0$ then Proposition 3.3 can be re-stated with $H(\beta)$ replaced by the $2mn\times 2mn$ matrix*

$$\begin{bmatrix} A_1^{-1}\otimes C & -A_1^{-1}B\otimes I_m \\ -C\otimes A_r^{-1} & I_m\otimes A_r^{-1}B \end{bmatrix} \times \begin{bmatrix} ((A+j\beta I_n)\oplus(-A+j\beta I_n))^{-1} & 0 \\ 0 & -I_{m^2}/(j2\beta) \end{bmatrix} \begin{bmatrix} -I_n\otimes B & B\otimes I_n \\ -C\otimes I_m & I_m\otimes C \end{bmatrix}$$

Note that the mild assumptions in Proposition 3.3 (*c*) allow a significant reduction in the dimension of the eigenvalue formulae for the critical integral gains. Indeed, $H(\beta)$ is of dimension $(2mn+m^2)\times(2mn+m^2)$ (or dimension $2mn\times 2mn$ if Corollary 3.4 applies) whereas the dimension of the generalized eigenvalue problem in Proposition 3.3 (*b*) is $(n+m)^2\times(n+m)^2$.

The application of Proposition 3.3 is now illustrated by two examples based on the systems considered in Examples 3.1 and 3.2.

Example 3.3: Consider the system $G(s) = (s+2)/(s+1)$ given in Example 3.1 and the horizontal band with $0<\beta\leqslant\sqrt{2}$. The critical integral gains are given by the gains at P and R in Fig. 4. By applying the standard calibration rule for root locus and expressing the gains at P and R in terms of β, we find that the critical integral gains are

$$3\mp 2(2-\beta^2)^{1/2} \tag{4}$$

Alternatively, using the formula in Proposition 3.3 (*a*), we find that

$$H(\beta) = \frac{1}{2\beta(1+4\beta^2)}\begin{bmatrix} 4\beta+j(4\beta^2-1) & -2\beta+j & 2\beta(1+j2\beta) \\ -2\beta-j & 4\beta-j(4\beta^2-1) & 2\beta(1-j2\beta) \\ 2\beta-j3 & 2\beta+j3 & 4\beta \end{bmatrix}$$

Applying a similarity transformation reveals that $H(\beta)$ has the same eigenvalues as

$$\hat{H}(\beta) = \frac{1}{\beta(1+4\beta^2)}\begin{bmatrix} 3\beta+j2(\beta^2-1) & j2 & 0 \\ -j2 & 3\beta-j2(\beta^2-1) & 0 \\ \beta-j3/2 & \beta+j3/2 & 0 \end{bmatrix}$$

The eigenvalues of $H(\beta)$ are therefore the eigenvalues of the upper-left 2×2 block of $\hat{H}(\beta)$ together with zero. Hence

$$\lambda_{\text{real}}(H(\beta)) = 0,\quad \frac{3\pm 2(2-\beta^2)^{1/2}}{1+4\beta^2} \tag{5}$$

Inverting the non-zero term in (5) and rationalizing the denominator gives (4) and hence parts (*a*) and (*c*) of Proposition 3.3 are analytically verified for $G(s) = (s+2)/(s+1)$. □

Example 3.4: Consider the five-state 3×3 system given in Example 3.2. Suppose that we wish to see how far we can increase integral action from zero whilst restricting the frequency of the oscillations to be below 2 rad s^{-1}. In other words, find $\hat{k}_I^{\max}$ for the $\beta = 2$ horizontal band. (Note that we have not yet enforced (asymptotic) stability of

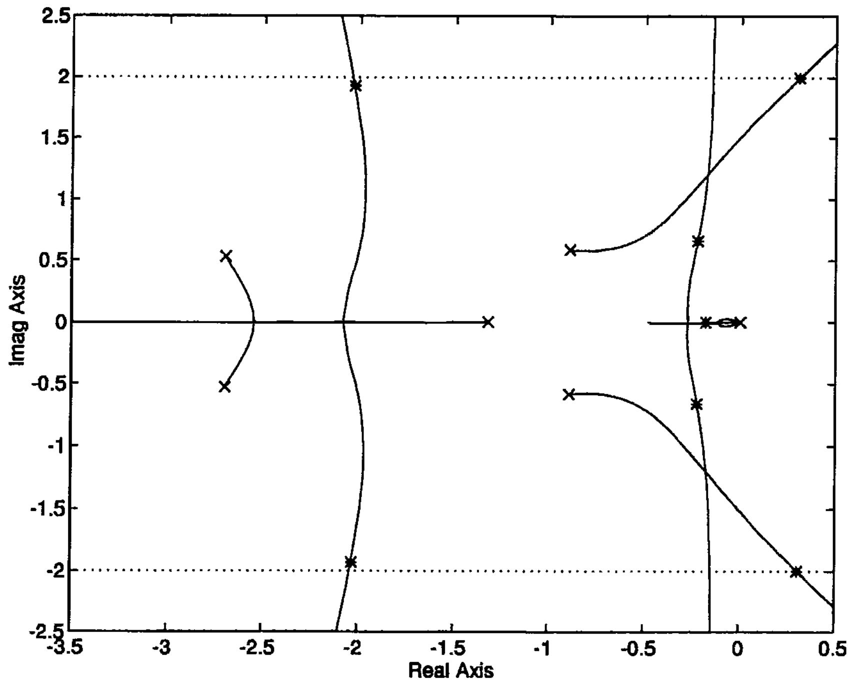

Figure 7. Integral gain root loci for Example 3.4, where $\times$ and $*$ denote the closed-loop eigenvalues for $k_I = 0$ and $k_I = \hat{k}_I^{max}$ respectively.

the closed loop.) Using Corollary 3.4, we find that $\hat{k}_I^{max} = 1{\cdot}5905$. To verify this calculation, the integral gain root loci of the system are plotted in Fig. 7. It can be seen that the closed-loop eigenvalues satisfy the oscillation frequency constraint for all $k_I \in (0, \hat{k}_I^{max})$, and that the oscillation frequency constraint is first violated for $k_I = \hat{k}_I^{max}$. However, it should be noted that two of the closed-loop eigenvalues move into the right half-plane before the oscillation frequency constraint is violated. The movement of eigenvalues into the right half-plane in this manner is obviously undesirable, and is the motivation for the next subsection. □

3.3. *Combining the regions*

Proposition 3(iv) of Saydy *et al.* (1990) shows that a guardian map for the intersection of two regions is the product of the guardian maps of the regions. This allows us to combine the results of Sections 3.1 and 3.2 as illustrated in the following example.

Example 3.5. Consider the system studied in Examples 3.2 and 3.4. Suppose that we wish to see how far we can increase integral action from zero whilst keeping the damping factor greater than 1/2 *and* restricting the frequency of the oscillations to be below 2 rad s^{-1}. In other words, find the minimum of the critical gains calculated in Examples 3.2 and 3.4. This cures the instability problem of Example 3.4. We get

$$\hat{k}_I^{max} = \min\{0{\cdot}1623, 1{\cdot}5905\} = 0{\cdot}1623$$

which is $\hat{k}_I^{max}$ from Example 3.2. Inspection of Fig. 5 in Example 3.2 verifies that both constraints are satisfied for all $k_I \in (0, 0{\cdot}1623)$ and that the first constraint violation occurs when $k_I = 0{\cdot}1623$.

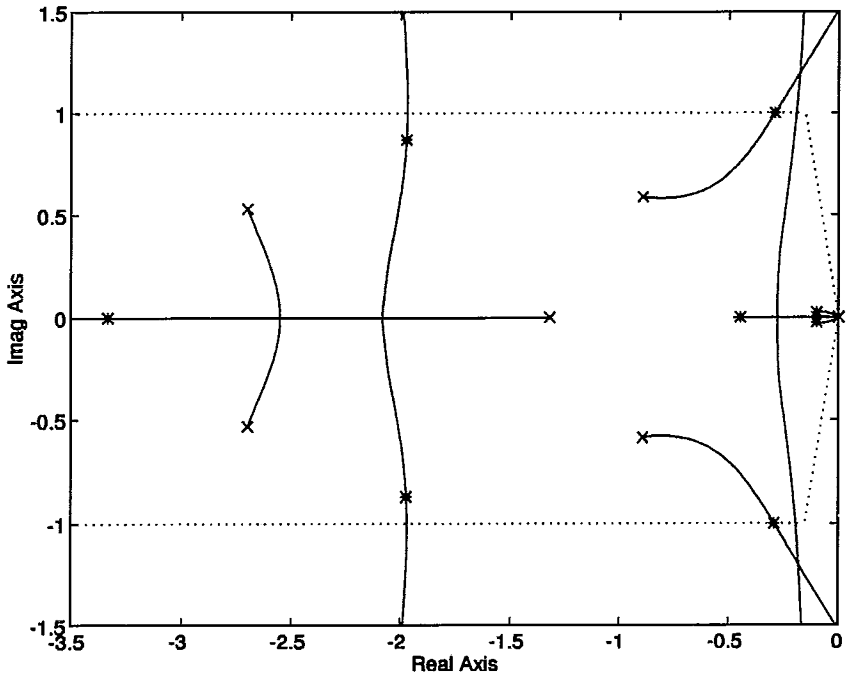

Figure 8. Integral gain root loci for the second part of Example 3.5, where $\times$ and $*$ denote the closed-loop eigenvalues for $k_{\mathrm{I}} = 0$ and $k_{\mathrm{I}} = 0{\cdot}2926$ respectively.

With the above constraints, the sector constraint is active. Of course, different values for the minimum damping factor and the maximum oscillation frequency may lead to the horizontal band constraint becoming active instead. For instance, if the constraints are to keep the damping factor greater than 15% and the oscillation frequency less than 1 rad s^{-1}, then

$$\hat{k}_{\mathrm{I}}^{\max} = \min\{0{\cdot}4406, 0{\cdot}2926\} = 0{\cdot}2926$$

and the oscillation frequency constraint is active, as shown in Fig. 8. □

We will not discuss the combined region further, as in each of the following sections it is clear that the sector and the band can be combined in a similar way to that described above.

4. Generalized proportional controllability

In this section we present results for proportional controllers that are analogous to the results in §3. In this and the following sections, we consider only strictly proper systems to ensure well-posedness of the closed loop. Given an $m \times m$ system $G(s) = C(sI - A)^{-1}B$ and the proportional controller $k_{\mathrm{P}} I_m$, the closed-loop dynamics matrix is

$$\bar{A}(k_{\mathrm{P}}) := A - k_{\mathrm{P}} BC \tag{6}$$

Since the eigenvalues of $\bar{A}$ move continuously with k_{P}, if A is in a region Γ then there exists a $\hat{k}_{\mathrm{P}}$ such that $\bar{A}$ is in Γ for all $k_{\mathrm{P}} \in [0, \hat{k}_{\mathrm{P}})$ (i.e. all $G(s)$ in Γ are 'Γ-proportional controllable'). We will use the term *radius of Γ-proportional controllability* to mean the largest value of $\hat{k}_{\mathrm{P}}$ such that $k_{\mathrm{P}} I_m$ Γ-assigns $G(s)$ for all $k_{\mathrm{P}} \in [0, \hat{k}_{\mathrm{P}})$ and will denote it by $\hat{k}_P^{max}$. The region to which $\hat{k}_P^{max}$ refers will be clear from the context. Note that in the single-input single-output case, if Γ is the left half-plane and $\hat{k}_P^{max} > 1$, then $\hat{k}_P^{max}$ is the gain margin of $G(s)$.

This section is kept short because the derivation of the formulae for the radius of Γ-proportional controllability (Propositions 4.1 (*a*) and 4.2 (*a*) below) is simpler than that for the radius of Γ-integral controllability. The simplification arises because $\bar{A}(k_P)$ in (6) is not block structured, in contrast to $\bar{A}(k_I)$ in (1). Indeed, the formulae for the radius of Γ-proportional controllability could have been derived from the modification of the results of Fu and Barmish (1988) to the region Γ.

4.1. *Proportional controllability in the left sector*

Consider the left sector defined in §3.1 and let $\lambda_i (1 \leqslant i \leqslant n)$ be the eigenvalues of $\bar{A}(k_P)$ in (6). A *critical proportional gain* (*for the left sector*) is defined to be any finite real value of k_P such that $e^{j\alpha}\lambda_p + e^{-j\alpha}\lambda_q = 0$, for some p, q $(1 \leqslant p,\ q \leqslant n)$. In the following proposition we give a formula for the radius of left-sector proportional controllability. We also give formulae for all the critical proportional gains, for later use.

Proposition 4.1: *Let* $G(s) = C(sI - A)^{-1} B$ *be an n-state* $m \times m$ *system.*

(*a*) *If* $G(s)$ *is in the left sector, then the radius of left-sector proportional controllability is given by*

$$\hat{k}_P^{max} = \frac{1}{\lambda_{max}^{+}(W(\alpha))}$$

where $W(\alpha)$ *is the* $n^2 \times n^2$ *matrix*

$$W(\alpha) := (e^{j\alpha} A \oplus e^{-j\alpha} A)^{-1} (e^{j\alpha} BC \oplus e^{-j\alpha} BC)$$

(*b*) *All the critical proportional gains for the left sector are given by*

$$\lambda_{real}(e^{j\alpha} A \oplus e^{-j\alpha} A, \quad e^{j\alpha} BC \oplus e^{-j\alpha} BC)$$

Proof: For the proof see Appendix D. □

4.2. *Proportional controllability in the horizontal band*

Consider the horizontal band defined in §3.2 and let λ_i $(1 \leqslant i \leqslant n)$ be the eigenvalues of $\bar{A}(k_P)$ in (6). A *critical proportional gain* (*for the horizontal band*) is defined to be any finite real value of k_P such that $\lambda_p - \lambda_q = j2\beta$, for some p, q $(1 \leqslant p,$ $q \leqslant n)$. In the following proposition we give a formula for the radius of horizontal-band proportional controllability. We also give formulae for all the critical proportional gains, for later use.

Proposition 4.2: *Let* $G(s) = C(sI - A)^{-1} B$ *be an n-state* $m \times m$ *system.*

(*a*) *If* $G(s)$ *is in the horizontal band, then the radius of horizontal-band proportional controllability is given by*

$$\hat{k}_P^{max} = \frac{1}{\lambda_{max}^{+}(X(\beta))}$$

where $X(\beta)$ *is the* $n^2 \times n^2$ *matrix*

$$X(\beta) := ((A + j\beta I_n) \oplus (-A + j\beta I_n))^{-1} (BC \oplus -BC)$$

(*b*) *All the critical proportional gains for the horizontal band are given by*

$$\lambda_{\text{real}}((A+j\beta I_n)\oplus(-A+j\beta I_n), BC\oplus -BC)$$

Proof: For the proof see Appendix E. □

5. Generalized PI controllability

In this section we extend the results on generalized integral and generalized proportional controllability to PI controllers. A system $G(s)$ in a region Γ is said to be *Γ-PI controllable* if there exist positive gains $\hat{k}_{\text{P}}$ and $\hat{k}_{\text{I}}$ such that the PI controller $(k_{\text{P}}+k_{\text{I}}/s)I_m$ Γ-assigns $G(s)$ for all $k_{\text{P}}\in[0,\hat{k}_{\text{P}})$ and $k_{\text{I}}\in(0,\hat{k}_{\text{I}})$. We will use the term *maximal region of Γ-PI controllability* to mean the largest region enclosing the origin in the positive quadrant of (k_{P}, k_{I}) gain-space that contains only gain pairs that Γ-assign $G(s)$. Where the region Γ is clear from the context, we will often refer to the maximal region of Γ-PI controllability as the *maximal region.*

Since all strictly proper systems in a region Γ are 'Γ-proportional controllable' (see §4), it is immediate that a strictly proper system in Γ is Γ-PI controllable if and only if it is Γ-integral controllable. This fact is the generalization to the region Γ of the corrected version of Lemma 10·4 of Lunze (1989).

Given an n-state $m\times m$ strictly proper system $G(s) = C(sI-A)^{-1}B$ in Γ, which is Γ-PI controllable, the results of §§3 and 4 can be used to find the maximal region for $G(s)$, as shown below. First, note that the closed-loop dynamics matrix formed by the negative unity feedback connection of the PI controller $(k_{\text{P}}+k_{\text{I}}/s)I_m$ to $G(s)$ is

$$\bar{A}(k_{\text{P}},k_{\text{I}}) := \begin{bmatrix} A-k_{\text{P}}BC & k_{\text{I}}B \\ -C & 0 \end{bmatrix}$$

It is simple to show that $\bar{A}(k_{\text{P}},k_{\text{I}})$ is equal to the closed-loop dynamics matrix formed by the negative unity feedback connection of the integral controller $k_{\text{I}}I_m/s$ to $G_{\text{P}}(s)$, where

$$G_{\text{P}}(s) := G(s)(I_m+k_{\text{P}}G(s))^{-1} = C_{\text{P}}(sI_n-A_{\text{P}})^{-1}B_{\text{P}}$$

and

$$A_{\text{P}} := A-k_{\text{P}}BC, \quad B_{\text{P}} := B, \quad C_{\text{P}} := C$$

In a similar way, $\bar{A}(k_{\text{P}},k_{\text{I}})$ is also equal to the closed-loop dynamics matrix formed by the negative unity feedback connection of the proportional controller $k_{\text{P}}I_m$ to $G_{\text{I}}(s)$, where

$$G_{\text{I}}(s) := G(s)(I_m+k_{\text{I}}G(s)/s)^{-1} = C_{\text{I}}(sI_{n+m}-A_{\text{I}})^{-1}B_{\text{I}}$$

and

$$A_{\text{I}} := \begin{bmatrix} A & k_{\text{I}}B \\ -C & 0 \end{bmatrix}, \quad B_{\text{I}} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_{\text{I}} := [C \quad 0]$$

The results of §§3 and 4 can be applied to $G_{\text{P}}(s)$ and $G_{\text{I}}(s)$ respectively to obtain the maximal region for $G(s)$ as shown in the following procedure.

Procedure 5.1: *Given a strictly proper system $G(s)$ in Γ that is Γ-*PI *controllable, the maximal region can be found in the following way.*

Step i. *Use the results of* §3 *to plot the positive critical integral gains* (*for the region Γ*) *of $G_{\text{P}}(s)$ for positive values of k_{P}.*

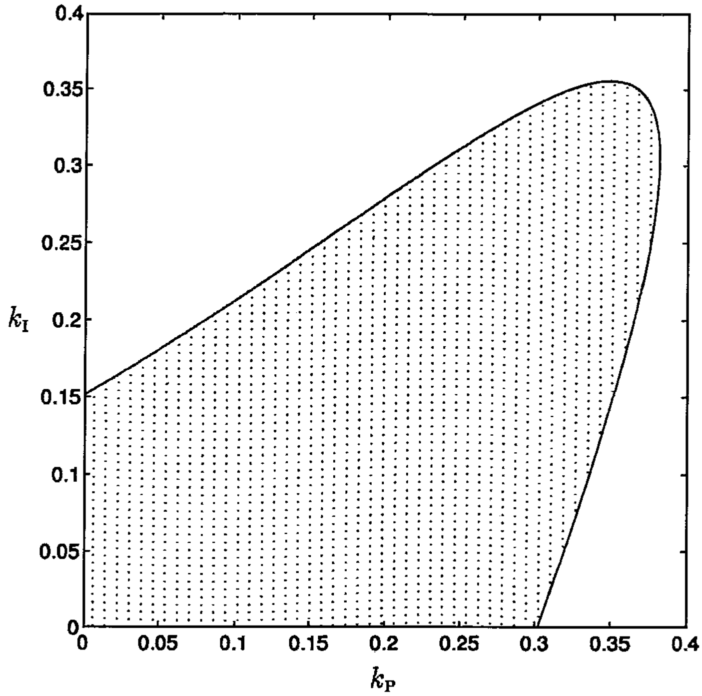

Figure 9. The maximal region (shaded) for the 45° left sector for Example 5.1.

Step ii. *Use the results of* § 4 *to plot the positive critical proportional gains* (*for the region* Γ) *of* $G_I(s)$ *for positive values of* k_I.

The maximal region is the region formed by Steps (i) *and* (ii) *which encloses the origin.* □

Remark 5.1: When Γ is a left sector, if $-G(0)$ is in the left sector then the simplified formulae in Proposition 3.2 (*c*) can be used in Procedure 5.1(i) for all positive values of k_P except the positive critical proportional gains (for the left sector) of $G(s)$. At these values of k_P (given in Proposition 4.1 (*b*)), Proposition 3.2 (*b*) can be used instead. Similarly, for the horizontal band, the simplified formulae in Proposition 3.3 (*c*) can be applied for all positive values of k_P except the positive critical proportional gains (for the horizontal band) of $G(s)$. At these values of k_P (given in Proposition 4.2 (*b*)), Proposition 3.3 (*b*) can be used instead. □

The calculation of the maximal region is illustrated in the following example for the left sector.

Example 5.1: Consider the single-input single-output system

$$G(s) = \frac{s^2 + 2s + 10}{s^3 + 5s^2 + 9s + 5}$$

and suppose that we wish to find how far we can increase the proportional and integral gains from zero whilst maintaining a damping factor greater than $1/\sqrt{2}$. In other words, find the maximal region for the 45° left sector. The maximal region was calculated using Procedure 5.1 and is shown in Fig. 9. □

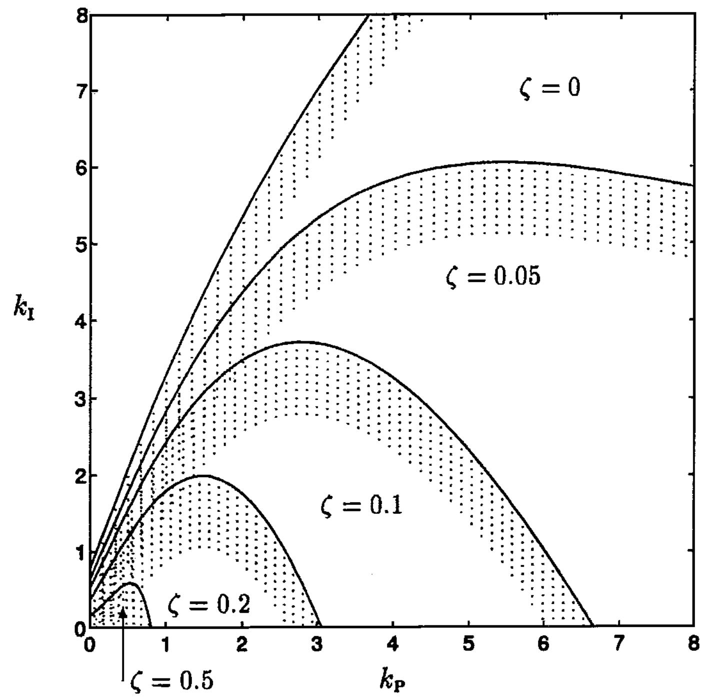

Figure 10. The maximal regions for various damping factors for Example 5.2.

In some application areas it may be of interest to determine the manner in which the size and shape of the maximal region vary with a parameter defining the stability region Γ. In general, it is difficult to make precise statements about the shape of the maximal region, as pointed out by Genesio and Tesi (1988). However, by using Procedure 5.1, the maximal region can be plotted easily and exactly for representative values of the parameter defining the stability region. The variation of the maximal region with the damping factor for a particular system is illustrated in the following example.

Example 5.2: Consider the five-state 3×3 system given in Example 3.2. The maximal regions (for the left sector) for damping factors of $\zeta = \sin \alpha = 0, 0{\cdot}05, 0{\cdot}1, 0{\cdot}2, 0{\cdot}5$ for this system were calculated using Procedure 5.1 and are shown in Fig. 10. □

An important issue in control system design is whether robustness against perturbations can be guaranteed. When the region Γ is the left half-plane, strong robustness results are available which are compatible with the approach taken in the present paper. In particular, it is immediate from the 16 Plant Theorem (Barmish *et al.* 1992) that the maximal region of left half-plane PI controllability for an interval plant family is the intersection of the maximal regions for 16 particular extreme plants known as the 'Kharitonov plants'. The maximal region for the interval plant family can therefore be easily calculated as the intersection of the maximal regions for the 0° sector for each of the 16 Kharitonov plants. Each such region can be computed directly using Procedure 5.1. See Mustafa (1994*d*) for an example. When Γ is a left-sector, we conjecture that a similar result to the 16 Plant theorem holds too. That would allow the maximal region of left-sector PI controllability for an interval plant family to be found as the intersection of maximal regions for a subset of distinguished extreme plants. Again, each such region could be computed directly using Procedure 5.1. Proof of such a conjecture may well follow from extensions of strong Kharitonov-type theorems for the left sector (Soh and Foo 1990, Foo and Soh 1989, Rantzer 1990).

6. Conclusions

A generalization of the concept of integral controllability to a left sector and to a horizontal band has been presented. Conditions for left-sector integral controllability were derived, generalizing those of Morari (1985), and it was shown that all systems in a horizontal band are horizontal-band integral controllable. By using block Kronecker algebra (Hyland and Collins 1989) to modify guardian maps (Saydy *et al.* 1990), closed eigenvalue formulae for the maximal low integral gain with respect to the sector and the band were derived, generalizing and improving on the results of Mustafa (1994*a*, 1994*c*). Closed eigenvalue formulae for the maximal low proportional gains for the sector and the band were also derived using similar techniques. It was shown how to use these results to find all low PI controller gains that place the closed-loop eigenvalues in the sector or the band (i.e. the maximal region in PI gain space). The method involved plotting the positive critical integral and proportional gains of certain auxiliary systems.

In the present paper attention has been focused on low positive gain controllers. Of course there may also be other (higher or negative) gain regions for which the closed-loop eigenvalues are in the desired region of the complex plane. As illustrated in Examples 3.1 and 3.3 and in Fig. 4, the critical gains also give the boundaries of all other gain regions for which the closed-loop eigenvalues are in the desired region of the complex plane. Hence, all such gain regions can be calculated using the techniques described here. See Mustafa (1994*d*) for the case of single-input single-output systems with the left half-plane as the desired region of the complex plane.

ACKNOWLEDGEMENTS

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Appendix A

Proof of Proposition 3.1: Recall the closed-loop dynamics matrix $\bar{A}(k_I)$ given in (1). It is well-known that the eigenvalues of $\bar{A}(k_I)$ vary continuously with k_I (see for example, p. 540 of Horn and Johnson 1985). Since $G(s)$ is in the left sector, the only eigenvalues of $\bar{A}(0)$ on the boundary of the left sector are those at the origin. So it is enough to consider just the zero eigenvalues of $\bar{A}(0)$ and to ensure that they move into the left sector for all sufficiently small positive k_I.

A formula for the derivatives of a certain class of multiple eigenvalues (which includes our case of interest) is given in the following lemma. The lemma is essentially Theorem 7 of Lancaster (1964), but is presented as a simplification of Theorem 4.1 of Sun (1990). Related results have also appeared in Kato (1966). Note that an eigenvalue is said to be *semi-simple* if the dimension of its associated eigenspace (i.e. its geometric multiplicity) is equal to its (algebraic) multiplicity (see for example, p. 41 of Kato 1966).

Lemma A.1: *Let $p \in \mathbb{R}$ and let $M(p)$ be an $n \times n$ matrix analytically dependent on p. Let $\lambda_1(p), \lambda_2(p), \ldots, \lambda_r(p)$ be the $r \leqslant n$ eigenvalues of $M(p)$ which coincide at λ_0 when $p = p_0$. If λ_0 is semi-simple then the derivatives of $\lambda_1(p)$, $\lambda_2(p)$, ..., $\lambda_r(p)$ at $p = p_0$ are the eigenvalues of*

$$Y_2^* \left.\frac{\partial M(p)}{\partial p}\right|_{p=p_0} X_2$$

where $X = [X_1\, X_2]$ and $Y = [Y_1\, Y_2]$, with $X, Y \in \mathbb{C}^{n \times n}$ and $X_2, Y_2 \in \mathbb{C}^{n \times r}$, are such that

$$Y^* M(p_0) X = \begin{bmatrix} \tilde{M} & 0 \\ 0 & \lambda_0 I_r \end{bmatrix}$$

where $Y^* X = I_n$ and λ_0 is not an eigenvalue of $\tilde{M}$.

Continuing the proof of Proposition 3.1, we note that

$$\bar{A}(0) = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}$$

Hence, using the notation of Lemma A.1 with $p = k_I$, $p_0 = 0$, $\lambda_0 = 0$ and $r = m$, we can choose

$$X = \begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_m \end{bmatrix} \quad \text{and} \quad Y^* = \begin{bmatrix} I_n & 0 \\ CA^{-1} & I_m \end{bmatrix} \tag{A 1}$$

where A is invertible because $G(s)$ is in the left sector by assumption. It is clear that the (multiple) eigenvalue of $\bar{A}(0)$ at the origin is semi-simple since the (2, 2)-blocks of X and Y in (A 1) are full rank. Applying Lemma A.1, the derivatives of the m zero eigenvalues of $\bar{A}(k_I)$ at $k_I = 0$ are the eigenvalues of

$$[CA^{-1}\ I_m] \begin{bmatrix} 0 & B \\ 0 & -D \end{bmatrix} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = CA^{-1}B - D = -G(0)$$

Hence, if the eigenvalues of $-G(0)$ lie in the left sector, then the m eigenvalues of $\bar{A}(0)$ at the origin move into the left sector for all sufficiently small positive k_I. Furthermore, if the m eigenvalues of $\bar{A}(0)$ at the origin move into the left sector for all sufficiently small positive k_I, then the eigenvalues of $-G(0)$ cannot be zero and must lie in the closed left sector. In the case where $m = 1$, the conditions collapse to the single condition $G(0) > 0$. □

Appendix B

Proof of Proposition 3.2: Recall the closed-loop dynamics matrix given in (1). Suppressing the explicit dependence of $\bar{A}$ on k_I, consider the function

$$v(k_I) = \det(e^{j\alpha}\bar{A} \oplus e^{-j\alpha}\bar{A})$$

The eigenvalues of $e^{j\alpha}\bar{A} \oplus e^{-j\alpha}\bar{A}$ are the pairwise sums of the eigenvalues of $e^{j\alpha}\bar{A}$ and $e^{-j\alpha}\bar{A}$. Thus, $v(k_I) = 0$ if and only if $e^{j\alpha}\bar{A} \oplus e^{-j\alpha}\bar{A}$ is singular if and only if $\bar{A}$ has a pair of eigenvalues λ_p, λ_q satisfying $e^{j\alpha}\lambda_p + e^{-j\alpha}\lambda_q = 0$. In particular, if $\bar{A}$ has all its eigenvalues in the closed left sector, $v(k_I) = 0$ if and only if $e^{j\alpha}\bar{A} \oplus e^{-j\alpha}\bar{A}$ is singular if and only if $\bar{A}$ has at least one eigenvalue on the boundary of the left sector. Hence $v(k_I)$ guards the (open) left sector in the sense of Saydy *et al.* (1990). It should be noted that the guardian map, $v(k_\mathrm{I})$, is similar to that given in Example 3.5 of Saydy *et al.* (1990), but has the advantages that it retains the block structure of $\bar{A}$ and that it is real for $\alpha = 0$. It is the structure that will be exploited to obtain the formulae.

Since $\bar{A}$ has a pair eigenvalues satisfying $e^{j\alpha}\lambda_p + e^{-j\alpha}\lambda_q = 0$ if and only if $v(k_\mathrm{I}) = 0$, the critical integral gains are the finite real solutions to $v(k_\mathrm{I}) = 0$. For the special case in part (*a*) of the proposition, left-sector integral controllability of $G(s)$ guarantees that for all sufficiently small positive k_I all the eigenvalues of $\bar{A}$ are in the (open) left sector. Hence, by construction of the guardian map, as k_I is increased, the first value for which at least one closed-loop eigenvalue is on the boundary of the left sector (i.e. $\hat{k}_\mathrm{I}^{\max}$) is the smallest positive real solution of $v(k_\mathrm{I}) = 0$ (or $+\infty$ if there are no finite positive real solutions). The remainder of the proof involves showing that the finite real solutions to $v(k_\mathrm{I}) = 0$ are given by the expressions in the proposition.

Firstly, note that $\bar{A}$ can be written as

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} - k_\mathrm{I} \begin{bmatrix} 0 & -B \\ 0 & D \end{bmatrix} \tag{B 1}$$

Using (B 1) and some block Kronecker algebra, $v(k_\mathrm{I})$ can be written as

$$v(k_\mathrm{I}) = \det\left(e^{j\alpha}\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \bar{\oplus}\, e^{-j\alpha}\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} - k_\mathrm{I}\left(e^{j\alpha}\begin{bmatrix} 0 & -B \\ 0 & D \end{bmatrix} \bar{\oplus}\, e^{-j\alpha}\begin{bmatrix} 0 & -B \\ 0 & D \end{bmatrix}\right)\right)$$

Hence, the finite real solutions to $v(k_\mathrm{I}) = 0$ (and therefore all the critical integral gains) are the finite real generalized eigenvalues given in part (*b*) of the proposition. We now turn to the proof of parts (*a*) and (*c*).

Using the definition of the block Kronecker sum

$$v(k_\mathrm{I}) = \det \begin{bmatrix} e^{j\alpha}A \oplus e^{-j\alpha}A & k_\mathrm{I} I_n \otimes e^{-j\alpha}B & k_\mathrm{I} e^{j\alpha}B \otimes I_n & 0 \\ -I_n \otimes e^{-j\alpha}C & e^{j\alpha}A \oplus (-k_\mathrm{I} e^{-j\alpha}D) & 0 & k_\mathrm{I} e^{j\alpha}B \otimes I_m \\ -e^{j\alpha}C \otimes I_n & 0 & (-k_\mathrm{I} e^{j\alpha}D) \oplus e^{-j\alpha}A & k_\mathrm{I} I_m \otimes e^{-j\alpha}B \\ 0 & -e^{j\alpha}C \otimes I_m & -I_m \otimes e^{-j\alpha}C & -k_\mathrm{I}(e^{j\alpha}D \oplus e^{-j\alpha}D) \end{bmatrix} \tag{B 2}$$

Since the assumptions of both parts (*a*) and (*c*) guarantee that $e^{j\alpha}A \oplus e^{-j\alpha}A$ is non-singular, we can apply the Schur formula for the determinant (see for example, p. 21 of Horn and Johnson 1985) to get

$$\begin{aligned} v(k_\mathrm{I}) &= \det(e^{j\alpha}A \oplus e^{-j\alpha}A) \\ &\times \det \left[\begin{bmatrix} e^{j\alpha}A \oplus (-k_\mathrm{I} e^{-j\alpha}D) & 0 & k_\mathrm{I} e^{j\alpha}B \otimes I_m \\ 0 & (-k_\mathrm{I} e^{j\alpha}D) \oplus e^{-j\alpha}A & k_\mathrm{I} I_m \otimes e^{-j\alpha}B \\ -e^{j\alpha}C \otimes I_m & -I_m \otimes e^{-j\alpha}C & -k_\mathrm{I}(e^{j\alpha}D \oplus e^{-j\alpha}D) \end{bmatrix} \right. \\ &\left. + k_\mathrm{I} \begin{bmatrix} I_n \otimes e^{-j\alpha}C \\ e^{j\alpha}C \otimes I_n \\ 0 \end{bmatrix} (e^{j\alpha}A \oplus e^{-j\alpha}A)^{-1} [I_n \otimes e^{-j\alpha}B \quad e^{j\alpha}B \otimes I_n \quad 0] \right] \end{aligned} \tag{B 3}$$

We now establish a Schur-type formula to simplify the second determinant in (B 3).

Lemma B.1: *Let $V, Z \in \mathbb{R}^{p\times p}$, $W \in \mathbb{R}^{p\times q}$, $X \in \mathbb{R}^{q\times p}$ and $Y \in \mathbb{R}^{q\times q}$. If V and $(Y-XV^{-1}W)$ are non-singular, then*

$$\det\begin{bmatrix} V+Z & W \\ X & Y \end{bmatrix} = \det(V)\det(Y-XV^{-1}W)$$

$$\times \det(I_q + (V^{-1}+V^{-1}W(Y-XV^{-1}W)^{-1}XV^{-1})Z)$$

Proof: We begin by noting that if V is non-singular

$$\begin{bmatrix} V & W \\ X & Y \end{bmatrix} = \begin{bmatrix} V & 0 \\ X & Y-XV^{-1}W \end{bmatrix}\begin{bmatrix} I_p & V^{-1}W \\ 0 & I_q \end{bmatrix}$$

Hence, if $(Y-XV^{-1}W)$ is non-singular,

$$\begin{bmatrix} V & W \\ X & Y \end{bmatrix}^{-1} = \begin{bmatrix} I_p & -V^{-1}X \\ 0 & I_q \end{bmatrix}\begin{bmatrix} V^{-1} & 0 \\ -(Y-XV^{-1}W)^{-1}XV^{-1} & (Y-XV^{-1}W)^{-1} \end{bmatrix}$$

Now

$$\det\begin{bmatrix} V+Z & W \\ X & Y \end{bmatrix} = \det\begin{bmatrix} V & W \\ X & Y \end{bmatrix}\det\left(\begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} + \begin{bmatrix} V & W \\ X & Y \end{bmatrix}^{-1}\begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}\right) \quad \text{(B 4)}$$

where use of the Schur formula gives

$$\det\begin{bmatrix} V & W \\ X & Y \end{bmatrix} = \det(V)\det(Y-XV^{-1}W) \quad \text{(B 5)}$$

Also note that

$$\det\left(\begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} + \begin{bmatrix} V & W \\ X & Y \end{bmatrix}^{-1}\begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} I_p + (V^{-1}+V^{-1}W(Y-XV^{-1}W)^{-1}XV^{-1})Z & 0 \\ * & I_q \end{bmatrix}\right)$$

$$= \det(I_p + (V^{-1}+V^{-1}W(Y-XV^{-1}W)^{-1}XV^{-1})Z) \quad \text{(B 6)}$$

where $*$ is unimportant for the proof. The formula is then obtained by combining (B 4), (B 5) and (B 6). □

Returning to the proof of parts (*a*) and (*c*) of Proposition 3.2, we simplify the second determinant in (B 3) by choosing (in the notation of Lemma B 1)

$$V = \begin{bmatrix} e^{j\alpha}A \otimes I_m & 0 \\ 0 & I_m \otimes e^{-j\alpha}A \end{bmatrix}$$

which is non-singular by assumption. This choice of V gives (again in the notation of Lemma B.1) $Y-XV^{-1}W = -k_I\,(e^{j\alpha}G(0) \oplus e^{-j\alpha}G(0))$ which is non-singular for $k_I \neq 0$

by assumption. By inspection of (9), $v(0) = 0$ so $k_I = 0$ is always a critical integral gain. For $k_I \neq 0$, applying Lemma B.1 gives

$$v(k_I) = (-k_I)^{m^2} f(\alpha) \det(I_{2nm} - k_I NM)$$

where

$$f(\alpha) = \det(A)^{2m} \det(e^{j\alpha}A \oplus e^{-j\alpha}A) \det(e^{j\alpha}G(0) \oplus e^{-j\alpha}G(0))$$

and

$$N = \begin{bmatrix} e^{-j\alpha}A^{-1} \otimes I_m & 0 \\ 0 & I_m \otimes e^{j\alpha}A^{-1} \end{bmatrix} + \begin{bmatrix} A^{-1}B \otimes I_m \\ I_m \otimes A^{-1}B \end{bmatrix} (e^{j\alpha}G(0) \oplus e^{-j\alpha}G(0))^{-1} [CA^{-1} \otimes I_m \quad I_m \otimes CA^{-1}]$$

$$M = \begin{bmatrix} I_n \otimes e^{-j\alpha}D & 0 \\ 0 & e^{j\alpha}D \otimes I_n \end{bmatrix} - \begin{bmatrix} I_n \otimes e^{-j\alpha}C \\ e^{j\alpha}C \otimes I_n \end{bmatrix} (e^{j\alpha}A \oplus e^{-j\alpha}A)^{-1} [I_n \otimes e^{-j\alpha}B \quad e^{j\alpha}B \otimes I_n]$$

Since $f(\alpha)$ is non-zero by assumption and is independent of k_I, all the finite real solutions to $v(k_I) = 0$ are the finite real solutions to $\det(I_{2mn} - k_I NM) = 0$ together with zero. The finite real solutions to $\det(I_{2mn} - k_I NM) = 0$ are simply the finite reciprocals of the real eigenvalues of NM. Part (*c*) of the proposition follows by multiplying out NM, collecting terms and simplifying using standard properties of Kronecker algebra to show that $NM = S(\alpha)$. The radius of left-sector integral controllability is the smallest positive real solution to $\det(I_{2mn} - k_I NM) = 0$ (or $+\infty$ if there are no finite positive real solutions). That is, $\hat{k}_I^{max} = 1/\lambda_{max}^{+}(NM)$, and hence part (*a*) of the proposition. □

Appendix C

Proof of Proposition 3.3: Using the expression for $\bar{A}$ given in (1), consider the function

$$v(k_I) = \det((\bar{A} + j\beta I_{n+m}) \bar{\oplus} (-\bar{A} + j\beta I_{n+m}))$$

Each eigenvalue of $(\bar{A} + j\beta I_{n+m}) \bar{\oplus} (-\bar{A} + j\beta I_{n+m})$ is equal to the difference between a pair of eigenvalues of $\bar{A}$ plus j2. Thus $v(k_I) = 0$ if and only if $(\bar{A} + j\beta I_{n+m}) \bar{\oplus} (-\bar{A} + j\beta I_{n+m})$ is singular if and only if $\bar{A}$ has a pair of eigenvalues λ_p, λ_q satisfying $\lambda_p - \lambda_q = j2\beta$. In particular, $v(k_I)$ guards the horizontal band and, in fact, is a block structured version of the guardian map of the horizontal band given in Example 3.4 of Saydy *et al.* (1990). Since the eigenvalues of the Kronecker and block Kronecker sums are equal, the modification retains the guarding property whilst exposing the structure of $\bar{A}$.

For parts (*b*) and (*c*) of the proposition, the critical integral gains are the finite real solutions to $v(k_I) = 0$. For the special case in part (*a*), since all systems in the horizontal band are horizontal-band integral controllable, it is immediate that $\bar{A}$ is in the horizontal band for all sufficiently small k_I. Thus (using a similar argument to that in the proof of Proposition 3.2 in Appendix B) as k_I is increased, the first value for which at least one closed-loop eigenvalue is on the boundary of the horizontal band (i.e. $\hat{k}_I^{max}$) is the smallest positive real solution of $v(k_I) = 0$ (or $+\infty$ if there are no finite positive real solutions). The remainder of the proof involves showing that the finite real solutions to $v(k_{\mathrm{I}}) = 0$ are given by the expressions in the proposition.

By using the expression for $\bar{A}$ in (8) and some block Kronecker algebra, $v(k_{\mathrm{I}})$ can be written as

$$v(k_{\mathrm{I}}) = \det\left(\begin{bmatrix} A+\mathrm{j}\beta I_n & 0 \\ -C & \mathrm{j}\beta I_m \end{bmatrix} \bar{\oplus} \begin{bmatrix} -A+\mathrm{j}\beta I_n & 0 \\ C & \mathrm{j}\beta I_m \end{bmatrix} - k_{\mathrm{I}}\left(\begin{bmatrix} 0 & -B \\ 0 & D \end{bmatrix} \bar{\oplus} \begin{bmatrix} 0 & B \\ 0 & -D \end{bmatrix}\right)\right)$$

Hence the finite real solutions to $v(k_{\mathrm{I}}) = 0$ (and therefore all the critical integral gains) are the finite generalized eigenvalues as given in part (*b*) of the proposition. We now turn to the proof of parts (*a*) and (*c*).

Using the definition of the block Kronecker sum

$$v(k_{\mathrm{I}}) = \det\left(\begin{bmatrix} (A+\mathrm{j}\beta I_n) \oplus (-A+\mathrm{j}\beta I_n) & -k_{\mathrm{I}} I_n \otimes B & k_{\mathrm{I}} B \otimes I_n & 0 \\ I_n \otimes C & (A+\mathrm{j}\beta I_n) \otimes (k_{\mathrm{I}} D + \mathrm{j}\beta I_m) & 0 & k_{\mathrm{I}} B \otimes I_m \\ -C \otimes I_n & 0 & (-k_{\mathrm{I}} D + \mathrm{j}\beta I_m) \oplus (-A+\mathrm{j}\beta I_n) & -k_{\mathrm{I}} I_m \otimes B \\ 0 & -C \otimes I_m & I_m \otimes C & (-k_{\mathrm{I}} D + \mathrm{j}\beta I_m) \oplus (k_{\mathrm{I}} D + \mathrm{j}\beta I_m) \end{bmatrix}\right)$$

Since $(A+\mathrm{j}\beta I_n) \oplus (-A+\mathrm{j}\beta I_n)$ is invertible under the assumptions of parts (*a*) and (*c*), we can apply the Schur formula to get

$$v(k_{\mathrm{I}}) = \det((A+\mathrm{j}\beta I_n) \oplus (-A+\mathrm{j}\beta I_n)) \det(L - k_{\mathrm{I}} M) \qquad \text{(C 1)}$$

where

$$L = \begin{bmatrix} (A+\mathrm{j}\beta I_n) \oplus \mathrm{j}\beta I_m & 0 & 0 \\ 0 & \mathrm{j}\beta I_m \oplus (-A+\mathrm{j}\beta I_n) & 0 \\ -C \otimes I_m & I_m \otimes C & \mathrm{j}2\beta I_{m^2} \end{bmatrix}$$

$$M = \begin{bmatrix} -I_n \otimes D & 0 & -B \otimes I_m \\ 0 & D \otimes I_n & I_m \otimes B \\ 0 & 0 & D \oplus (-D) \end{bmatrix} + \begin{bmatrix} I_n \otimes C \\ -C \otimes I_n \\ 0 \end{bmatrix} ((A+\mathrm{j}\beta I_n) \oplus (-A+\mathrm{j}\beta I_n))^{-1} [-I_n \otimes B \quad B \otimes I_n \quad 0] \qquad \text{(C 2)}$$

Note that $(A+\mathrm{j}\beta I_n) \oplus \mathrm{j}\beta I_m = A_1 \otimes I_m$, and similarly that $\mathrm{j}\beta I_m \oplus (-A+\mathrm{j}\beta I_n) = I_m \otimes A_{\mathrm{r}}$, where A_1 and A_{r} are defined in Proposition 3.3. The determinant of L can therefore be written as

$$\det(L) = (\mathrm{j}2\beta)^{m^2} \det(A_1)^m \det(A_{\mathrm{r}})^m$$

Since A_1 and A_{r} are non-singular by assumption, L is invertible. Thus

$$v(k_{\mathrm{I}}) = f(\beta) \det(I_{2mn+m^2} - k_{\mathrm{I}} L^{-1} M)$$

where

$$f(\beta) = (\mathrm{j}2\beta)^{m^2} \det(A_1)^m \det(A_{\mathrm{r}})^m \det((A+\mathrm{j}\beta I_n) \oplus (-A+\mathrm{j}\beta I_n))$$

and

$$L^{-1} = \begin{bmatrix} A_1^{-1} \otimes I_m & 0 & 0 \\ 0 & I_m \otimes A_r^{-1} & 0 \\ (CA_1^{-1} \otimes I_m)/(j2\beta) & -(I_m \otimes CA_r^{-1})/(j2\beta) & I_{m^2}/(j2\beta) \end{bmatrix}$$

The assumptions ensure that $f(\beta) \neq 0$, and hence all the finite real solutions to $v(k_I) = 0$ are the finite real solutions to $\det(I_{2mn+m^2} - k_I L^{-1}M) = 0$ (i.e. the finite reciprocals of the real eigenvalues of $L^{-1}M$). By collecting terms and simplifying using properties of Kronecker algebra, the matrix inversion lemma (see for example, p. 19 of Horn and Johnson 1985) and the facts that $G(j2\beta) = D + CA_r^{-1}B$ and $G(-j2\beta) = D - CA_1^{-1}B$, it follows that $L^{-1}M = H(\beta)$, and hence part (*c*). For part (*a*), the radius of horizontal-band integral controllability is the smallest positive real solution to $\det(I_{2mn+m^2} - k_I L^{-1}M) = 0$ (or $+\infty$ if there are no finite positive real solutions). Hence, $k_I^{\max} = 1/\lambda_{\max}^{+}(L^{-1}M)$. □

Appendix D

Proof of Proposition 4.1: Recall the closed-loop dynamics matrix given in (6) and consider the function

$$v(k_P) = \det(e^{j\alpha}\bar{A} \oplus e^{-j\alpha}\bar{A})$$

By similar arguments to those in the proof of Proposition 3.2 in Appendix B, for part (*b*) we require all the finite real solutions to $v(k_P) = 0$ and for part (*a*) we require the smallest positive real solution (or $+\infty$ if there are no finite positive real solutions). Part (*b*) follows by writing

$$v(k_P) = \det(e^{j\alpha}A \oplus e^{-j\alpha}A - k_P(e^{j\alpha}BC \oplus e^{-j\alpha}BC))$$

The assumptions in part (*a*) ensure that $e^{j\alpha}A \oplus e^{-j\alpha}A$ is non-singular and hence part (*a*) follows by rewriting $v(k_P)$ as

$$v(k_P) = \det(e^{j\alpha}A \oplus e^{-j\alpha}A)\det(I - k_P W(\alpha))$$ □

Appendix E

Proof of Proposition 4.2: Recall the closed-loop dynamics matrix given in (6) and consider the function

$$v(k_P) = \det((\bar{A} + j\beta I_n) \oplus (-\bar{A} + j\beta I_n))$$

By similar arguments to those in the proof of Proposition 3.3 in Appendix C, for part (*b*) we require all the finite real solutions to $v(k_P) = 0$ and for part (*a*) we require the smallest positive real solution (or $+\infty$ if there are no finite positive real solutions). Part (*b*) follows by writing

$$v(k_P) = \det((A + j\beta I_n) \oplus (-A + j\beta I_n) - k_P(BC \oplus -BC))$$

The assumptions in part (*a*) ensure that $(A + j\beta I_n) \oplus (-A + j\beta I_n)$ is non-singular and hence part (*a*) follows by rewriting $v(k_P)$ as

$$v(k_P) = \det((A + j\beta I_n) \oplus (-A + j\beta I_n))\det(I - k_P X(\beta))$$ □

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