Performance of Wavelet Packet-Division Multiplexing in Impulsive and Gaussian Noise

K. Max Wong, Jiangfeng Wu, Timothy N. Davidson, Qu Jin, and P.-C. Ching

***Abstract*—Wavelet packet-division multiplexing (WPDM) is a high-capacity, flexible, and robust multiple-signal transmission technique in which the message signals are waveform coded onto wavelet packet basis functions for transmission. In this letter, we derive an expression for the probability of error for a WPDM scheme in the presence of both impulsive and Gaussian noise sources and demonstrate that WPDM can provide greater immunity to impulsive noise than both a time-division multiplexing scheme and an orthogonal frequency-division multiplexing scheme.**

***Index Terms*—Impulse noise, multiuser communication, wavelet packet-division multiplexing.**

I. INTRODUCTION

ORTHOGONAL waveform coding has been widely used for *multiplexing* [1] in the form of frequency-division multiplexing (FDM) or time-division multiplexing (TDM). However, the recently developed wavelet packet decompositions generate a set of self and mutually orthogonal waveforms which could also be used for (synchronous) orthogonal multiplexing [2]. Whilst all synchronous orthogonal multiplexing schemes perform identically in additive white Gaussian noise (AWGN), they may perform differently in impulsive noise [3]. Impulsive noise is a primary source of performance degradation in several applications, including data transmission over telephone networks, and its effects on various digital communication schemes have received considerable attention; e.g., [4]–[7]. In this letter, we demonstrate that wavelet packet-division multiplexing (WPDM) [8]–[12] can provide a substantially greater immunity to impulsive noise than both TDM and orthogonal frequency-division multiplexing/multi-carrier modulation (OFDM-MCM).

First, let us briefly review WPDM. Let $g_0[n]$ be a length L finite impulse response filter which is self-orthogonal at even translations, and let $g_1[n] = (-1)^n g_0[L-1-n]$. Under some mild conditions [13], we can obtain a function $\phi_{01}(t) = \sqrt{2}\sum_n g_0[n]\phi_{01}(2t - nT_0)$, for a given interval T_0. Using $g_0[n]$, $g_1[n]$, and $\phi_{01}(t)$, we can then define a family of functions $\phi_{\ell m}(t), \ell \geq 0, 1 \leq m \leq 2^\ell$, in a binary tree structure, with the subscripts denoting the "level" of a node in the tree and its position within that level, respectively. The functions at the *terminals* of the tree form a *wavelet packet* [13]. They are self and mutually orthogonal at integer multiples of $T_\ell = 2^\ell T_0$ and have a finite duration $D_\ell = (L-1)T_\ell$. In WPDM [12], the binary messages $\sigma_{\ell m}[n] = \pm 1$ are waveform coded by pulse amplitude modulation of $\phi_{\ell m}(t - nT_\ell)$ and are then added together to form the composite signal $s(t)$. By exploiting the wavelet packet tree structure, WPDM can be implemented using a transmultiplexer and a single modulator, as illustrated in Fig. 1. In that figure

$$s(t) = \sum_k \sigma_{01}[k]\phi_{01}(t - kT_0) \tag{1}$$

where $\sigma_{01}[k] = \sum_{(\ell,m)\in\mathcal{T}}\sum_n f_{\ell m}[k - 2^\ell n]\sigma_{\ell m}[n]$, with $\mathcal{T}$ being the set of terminal index pairs and $f_{\ell m}[k]$ the equivalent filter from the (ℓ, m)th terminal to the root of the tree. The original messages can be recovered from $\sigma_{01}[k]$ using $\sigma_{\ell m}[n] = \sum_k f_{\ell m}[k - 2^\ell n]\sigma_{01}[k]$.

We can view WPDM as a combination of TDM and FDM. The coding waveforms overlap in *both* time and frequency, but orthogonality is preserved. Since we do not require "guard bands" nor "guard times" to ensure orthogonality in a practical system, it is possible to increase the number of users sharing a given channel over that of conventional FDM and TDM [12]. We can also interpret WPDM as generalized orthogonal code-division multiplexing (CDM) in which the "codes" are the equivalent filters $f_{\ell m}[k]$ and the "chip" waveform is $\phi_{01}(t)$. This class of generalized CDM includes conventional orthogonal CDM (i.e., Walsh–Hadamard CDM), but extends those schemes to allow for real-valued orthogonal codes that overlap in time and orthogonal chip waveforms that have a duration longer than the chip interval.

II. ANALYSIS OF IMPULSIVE NOISE EFFECTS ON WPDM

Consider the model of a WPDM receiver illustrated in Fig. 2. The demodulated signal is $r(t) = \sqrt{E}s(t) + \xi(t) + \sqrt{E}\sum_i a_i \cos\theta_i\, h_p(t - \tau_i)$, where $\xi(t)$ is the real part of the envelope of the bandpass (BP) Gaussian channel noise $\xi_c(t)$ at the output of the BP channel filter, and a_i, θ_i, and τ_i are the normalized amplitude, phase, and arrival time of the ith impulsive noise burst, respectively. Here, $h_p(t)$ is the real part of the envelope of the impulse response of the

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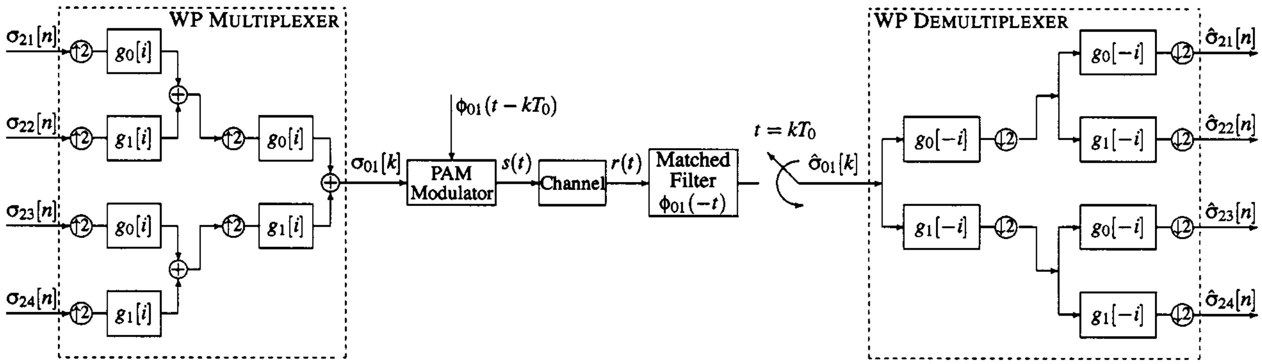

Fig. 1. The transmultiplexer implementation of a four-user WPDM scheme.

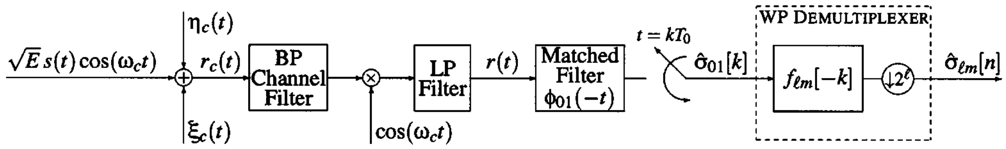

Fig. 2. Receiver model for the $(\ell,\, m)$th terminal with bandpass Gaussian and impulsive noise sources $\xi_c(t)$ and $\eta_c(t)$, respectively.

BP channel filter, which has unit zero-frequency (DC) gain. We will assume that the binary data symbols are independent and equally likely, and that:

1) $\{\tau_i\}_{i\in\mathbb{Z}}$ is a set of Poisson points with an average arrival rate ν such that $\nu D_\ell \ll 1$ for all terminals.
2) a_i, θ_i, τ_i, and $\xi_c(t)$ are independent; $\xi_c(t)$ is zero-mean, stationary, white, and Gaussian with power spectral density $N_0/2$; θ_i are independent and identically distributed (i.i.d.) uniformly on $[0,\, 2\pi)$; a_i are i.i.d. and positive with a given distribution.

Since $\phi_{\ell m}(t)$ is zero outside $(0,\, D_\ell)$, for an impulsive noise burst to significantly affect $\hat{\sigma}_{\ell m}[n]$, it must arrive within $\mathcal{D}_{\ell m;\,n} \simeq (nT_\ell,\, nT_\ell + D_\ell)$. Therefore, Assumption 1) implies that the probability of that more than one impulsive noise burst affects a given bit is negligible. Hence

$$\hat{\sigma}_{\ell m}[n] = \sigma_{\ell m}[n] + \xi_{\ell m}[n] + \beta_{\ell m}[n]\, a\, \cos\,\theta\, q_{\ell m}(\tau - nT_\ell) \tag{2}$$

where the Gaussian noise at the (ℓ, m)th terminal is $\xi_{\ell m}[n] = \sum_k f_{\ell m}[k - 2^\ell n]\xi_{01}[k]$, with $\xi_{01}[k]$ being the demodulated Gaussian channel noise, $\xi_{01}[k] = (1/\sqrt{E}) \int \xi(t)\phi_{01}(t - kT_0)\, dt$. For the impulsive noise component, $q_{\ell m}(\tau) = \sum_k f_{\ell m}[k] \int h_p(t-\tau)\phi_{01}(t)\, dt$, and $\beta_{\ell m}[n] = 1$ if $\exists i$ such that $\tau_i \in \mathcal{D}_{\ell m;\,n}$, and is zero otherwise. The quantities a and θ inherit the distributions of a_i and θ_i, respectively, and Assumption 1) implies that τ is uniformly distributed on $\mathcal{D}_{\ell m;\,n}$ [14]. Using the orthonormality of $f_{\ell m}[k]$ and $\phi_{01}(t)$, $\xi = \xi_{\ell m}[n]$ is a zero-mean Gaussian random variable independent of n, ℓ, and m, with variance N_0/E. Hence, the probability of error in $\hat{\sigma}_{\ell m}[n]$ is

$$\begin{aligned} P_{\ell m;\,n}(e) = {} & P(\beta_{\ell m}[n] = 0) P_{\ell m;\,n}(e|\beta_{\ell m}[n] = 0) \\ & + P(\beta_{\ell m}[n] = 1) P_{\ell m;\,n}(e|\beta_{\ell m}[n] = 1) \end{aligned} \tag{3}$$

where $P_{\ell m;\,n}(e|\beta_{\ell m}[n] = 0) = (1/2)\mathrm{erfc}(\sqrt{E/2N_0})$ [1], and under Assumption 1), $P(\beta_{\ell m}[n] = 0) \approx 1 - \nu D_\ell$ and $P(\beta_{\ell m}[n] = 1) \approx \nu D_\ell$ [14]. Since the parameters a, θ, τ, and ξ are independent

$$\begin{aligned} & P_{\ell m;\,n}(e|\beta_{\ell m}[n] = 1) \\ & = \iiiint P_{\ell m;\,n}(e|\beta_{\ell m}[n] = 1,\, a,\, \tau,\, \theta,\, \xi) \\ & \quad \cdot p_\Xi(\xi)\, d\xi\, p_\Theta(\theta)\, d\theta\, p_T(\tau)\, d\tau\, p_A(a)\, da \end{aligned} \tag{4}$$

where $p_X(x)$ denotes the probability density function (pdf) of the variable x. Assuming that $\sigma_{\ell m}[n] = 1$, without loss of generality, $P_{\ell m;\,n}(e|\beta_{\ell m}[n] = 1,\, a,\, \tau,\, \theta,\, \xi) = 1$ if $1 + \xi + a\,\cos\theta\, q_{\ell m}(\tau - nT_\ell) < 0$, and is zero otherwise. Evaluating the first two inner integrals in (4) and substituting into (3), we have [15]

$$\begin{aligned} & P_{\ell m;\,n}(e) \\ & = \frac{1 - \nu D_\ell}{2}\mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) + \frac{\nu}{2} \\ & \quad \cdot \int_0^\infty \int_0^{D_\ell} \mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}(1 - a q_{\ell m}(\tau))\right) d\tau \\ & \quad \cdot p_A(a)\, da - \frac{\nu}{\pi\sqrt{\pi}}\sqrt{\frac{E}{2N_0}} \int_0^\infty \int_0^{D_\ell} \int_{-1}^1 \cos^{-1}\gamma \\ & \quad \cdot \exp\left(-\frac{E(1 + a q_{\ell m}(\tau)\gamma)^2}{2N_0}\right) d\gamma\, q_{\ell m}(\tau)\, d\tau\, a p_A(a)\, da \end{aligned} \tag{5}$$

where $\cos^{-1}\gamma \in [0,\, \pi]$. Equation (5) is independent of n and will be denoted by $P_{\ell m}(e)$. Since the terminals at different levels have different bit rates, $1/T_\ell$, the overall probability of error is $\overline{P}(e) \triangleq \sum_{(\ell, m)\in\mathcal{T}} 2^{-\ell} P_{\ell m}(e)$.

The receiver impulse characteristic (RIC) [4] for the $(\ell,\, m)$th terminal, denoted by $R_{\ell m}(a)$, is defined so that $P_{\ell m}(e|\xi = 0) = \nu \int_0^\infty R_{\ell m}(a) p_A(a)\, da$. Hence, it captures the robustness of a given scheme to an impulse of a given amplitude in the absence

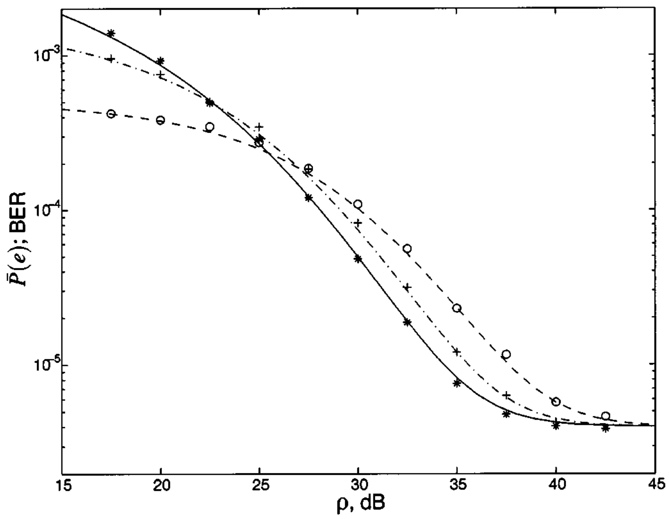

Fig. 3. Calculated probability of error $\overline{P}(e)$ (curves) and simulated BER (points) against SImpNR ρ for the scenario in Example 1. Solid and asterisk: WPDM; dashed and circle: TDM; dashed-dot and plus: OFDM-MCM.

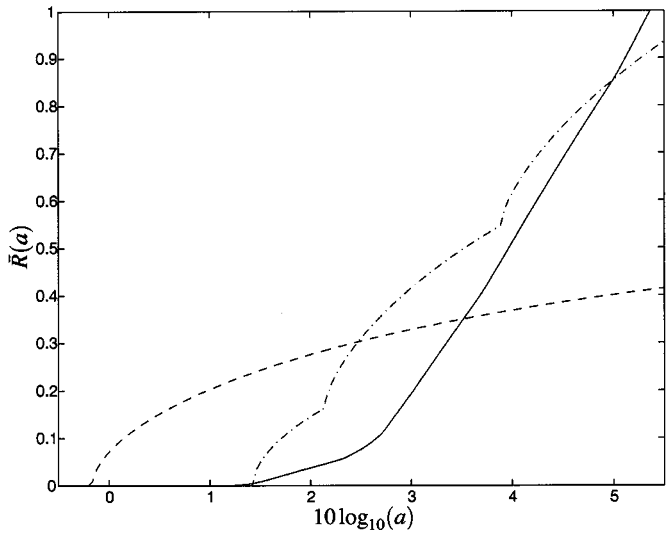

Fig. 4. The (average) receiver impulse characteristic $\overline{R}(a)$ for the schemes in Example 1. Solid: WPDM; dashed: TDM; dashed-dot: OFDM-MCM.

of Gaussian noise. An expression for the RIC can be easily obtained from (4) [15]. The overall RIC, $\overline{R}(a)$, can be defined in a similar manner to $\overline{P}(e)$ above.

III. WPDM Performance Comparisons

We now compare the performance of WPDM to that of TDM and a real-valued OFDM-MCM scheme. The composite signal for TDM and OFDM-MCM can be written in the form of (1) as $s(t) = \sum_k \sigma_0 p_0(t - kT_0)$, where, for M users, $\sigma_0[k] = \sum_{m=1}^{M} f_m[k - Mn]\sigma_m[n]$. For TDM, $f_m[k] = \delta[k - m + 1]$, and for the real-valued OFDM-MCM scheme, they are the synthesis filters from a discrete cosine transform; e.g., $f_m[k] = \sqrt{2/M}\ \cos((\pi(2k+1)(2m-1))/4M)$, for $k = 0, 1, \cdots, M-1$. We choose $p_0(t)$ to be the unit-energy rectangular function on $(0, T_0)$. With the minor modifications suggested by the equivalent filters above, the analysis in Section II also applies to TDM and OFDM-MCM [15].

Example 1: Consider the transmission of binary data from four message sources, each with a bit rate $1/T_2$. The output of the distortionless channel is corrupted by AWGN and impulsive noise bursts with an average arrival rate of $\nu T_0 = 10^{-3}$ and amplitudes a from a log-normal pdf [4] with *skewness* $20\log_{10}(\sqrt{\mathbb{E}\{a^2\}}/\mathbb{E}\{a\}) = 1$, where $\mathbb{E}\{\cdot\}$ denotes expectation. The BP channel filter has an LP envelope $h_p(t) = \exp(-t/C)/C$ for $t \geq 0$,with $C = T_0/20$. The message sources were multiplexed using: a) WPDM with a standard Daubechies filter [13] of length 14; b) TDM; or c) the real-valued OFDM-MCM scheme outlined above. For a fixed signal-to-Gaussian-noise ratio of $E/N_0 = 13$ dB, the overall probabilities of error for these multiplexing schemes were evaluated as described in Section II and plotted (Fig. 3) against signal-to-impulsive-noise ratio (SImpNR), $\rho = 1/(\nu T_0 \mathbb{E}\{a^2\})$. Computer simulations of these schemes, which are in close agreement with the analysis, are also shown in Fig. 3. It can be observed that WPDM provides greater immunity to impulsive noise than TDM when the SImpNR is over 26 dB. This is because the WPDM waveforms overlap in time, and hence the energy of an impulsive noise burst is dispersed over several bits at each terminal. Therefore, a moderate noise burst which is strong enough to cause an error in one bit in TDM may be sufficiently dispersed in WPDM so as not to cause an error. This advantage is clearly indicated in the superior RIC of WPDM for moderate amplitudes (Fig. 4). Similar advantages over TDM have been observed for OFDM-MCM [7], but since WPDM waveforms from the same terminal overlap with each other, whereas OFDM-MCM waveforms do not, the dispersion of the noise bursts is greater in WPDM. Hence, the superior performance of WPDM at moderate SImpNR's (Figs. 3 and 4). At high SImpNR's, the probability of error is dominated by the effects of the Gaussian noise, and hence the performance of all orthogonal multiplexing schemes is the same. At very low SImpNR's, however, the performance of WPDM degrades with respect to that of TDM because a strong noise burst may induce more than one bit error in WPDM, whereas it can induce at most one error in TDM. Similar performance degradation has also been observed for OFDM-MCM [7]. (Saturating receivers [4] provide protection from large noise bursts and may improve the performance of WPDM and OFDM-MCM at very low SImpNR's.) □

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