Solutions to Homework Assignment #2

1. The impulse response of an LTI system is shown in Fig. 1 below. Determine and carefully sketch the response of this system to the input $x[n] = u[n-4]$. (10 pts)

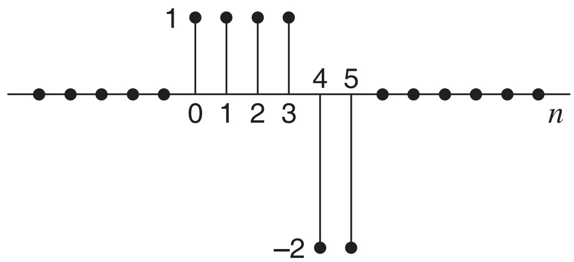


Figure 1: Impulse response $h[n]$ for Problem #1.

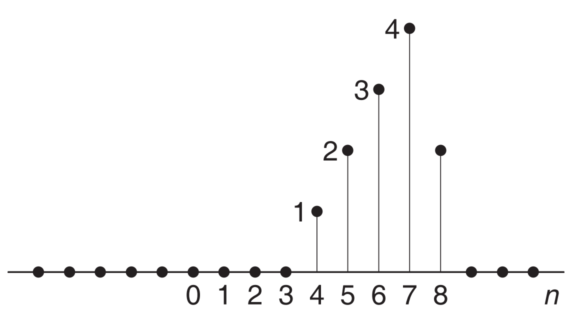
The output of this system $y[n] = h[n]*x[n]$, where $*$ indicates the convolution summation operation. For the input $x[n] = u[n-4]$, the output $y[n]$ is described by:

$$\begin{aligned} y[n] &= x[n]*h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} u[k-4]h[n-k] \\ &= \sum_{k=4}^{\infty} h[n-k]. \end{aligned}$$

Evaluating the above summation gives:

For $n < 4$: $y[n] = 0$
For $n = 4$: $y[n] = h[0] = 1$
For $n = 5$: $y[n] = h[1] + h[0] = 2$
For $n = 6$: $y[n] = h[2] + h[1] + h[0] = 3$
For $n = 7$: $y[n] = h[3] + h[2] + h[1] + h[0] = 4$
For $n = 8$: $y[n] = h[4] + h[3] + h[2] + h[1] + h[0] = 2$
For $n \geq 9$: $y[n] = h[5] + h[4] + h[3] + h[2] + h[1] + h[0] = 0.$

This sequence is shown graphically below:



2. **Consider the composite system shown in Fig. 2 below.**
   a. **Find the impulse response $h[n]$ of the overall system.**
   b. **Find the frequency response of the overall system. (Note: Tables of Fourier transform properties and Fourier transform pairs are given on the last page.)**
   c. **Specify a difference equation that relates the output $y[n]$ to the input $x[n]$.**
   d. **Is this system causal? Under what conditions would the system be stable?** **(40 pts)**

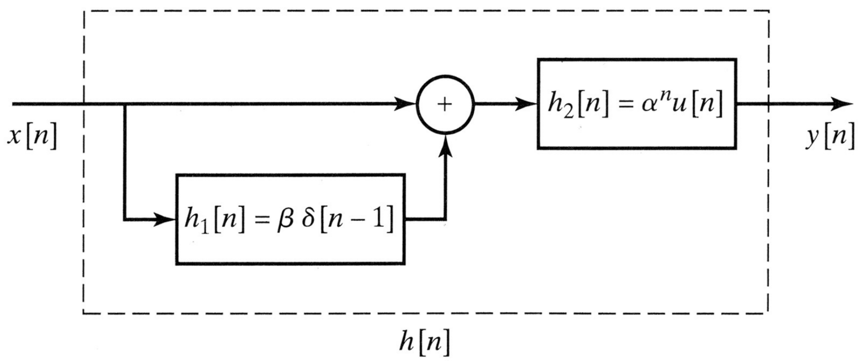


Figure 2: Composite system for Problem #2.

a. From Fig. 2:

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= x[n] * (\delta[n] + h_1[n]) * h_2[n] \\ &= x[n] * h[n], \end{aligned}$$

where the overall impulse response $h[n]$ is given by:

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \beta\alpha^{n-1} u[n-1]. \end{aligned}$$

b. Making use of Fourier Transform Pair 4. of Table 2.3 and Fourier Transform Property 2. of Table 2.2, the Fourier transform of $h[n]$ from part a. is:

$$\begin{aligned} H(e^{j\omega}) &= \mathcal{F}\{\alpha^n u[n] + \beta\alpha^{n-1} u[n-1]\} \\ &= \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \quad \text{for } |a| < 1. \end{aligned}$$

c. The simpler method of deriving an LCCD equation to describe the overall system is to make use of the definition of the frequency response:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})},$$

from which we obtain:

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} X(e^{j\omega})$$
$$\Rightarrow [1-\alpha e^{-j\omega}]Y(e^{j\omega}) = [1+\beta e^{-j\omega}]X(e^{j\omega})$$
$$\Rightarrow Y(e^{j\omega}) - \alpha e^{-j\omega} Y(e^{j\omega}) = X(e^{j\omega}) + \beta e^{-j\omega} X(e^{j\omega}).$$

Taking the inverse Fourier transform of both sides of the equation above results in the LCCD equation:

$$y[n] - \alpha y[n-1] = x[n] + \beta x[n-1].$$

The more complicated method is to evaluate the convolution directly in the time domain:

$$\begin{aligned}
y[n] &= h[n] * x[n] \\
&= (\alpha^n u[n] + \beta \alpha^{n-1} u[n-1]) * x[n] \\
&= (\alpha^n u[n]) * x[n] + (\beta \alpha^{n-1} u[n-1]) * x[n] \\
&= \sum_{k=-\infty}^{\infty} \alpha^k u[k] x[n-k] + \beta \sum_{k=-\infty}^{\infty} \alpha^{k-1} u[k-1] x[n-k] \\
&= \sum_{k=0}^{\infty} \alpha^k x[n-k] + \beta \sum_{k=1}^{\infty} \alpha^{k-1} x[n-k] \\
&= \sum_{k=0}^{\infty} \alpha^k x[n-k] + \beta \sum_{r=0}^{\infty} \alpha^r x[n-r-1] \quad (\text{where } r = k-1) \\
&= \sum_{k=0}^{\infty} \alpha^k x[n-k] + \beta \alpha^k x[n-k-1]. \quad (\text{where } k = r)
\end{aligned}$$

In order to obtain an LCCD equation, we observe that:

$$\begin{aligned}
\alpha y[n-1] &= \alpha \sum_{l=0}^{\infty} \alpha^l x[n-l-1] + \beta \alpha^l x[n-l-2] \quad (\text{where } l = k) \\
&= \sum_{l=0}^{\infty} \alpha^{l+1} x[n-l-1] + \beta \alpha^{l+1} x[n-l-2] \\
&= \sum_{k=1}^{\infty} \alpha^k x[n-k] + \beta \alpha^k x[n-k-1] \quad (\text{where } k = l+1)
\end{aligned}$$

$$\Rightarrow y[n] - \alpha y[n-1] = \alpha^0 x[n-0] + \beta\, \alpha^0 x[n-0-1]$$

$$\Rightarrow y[n] - \alpha y[n-1] = x[n] + \beta x[n-1],$$

which is identical to the LCCD equation obtained using the Fourier method above.

d. From part a., $h[n] = 0$ for all $n < 0 \Rightarrow$ the overall system is causal. If the system is stable, then its Fourier transform exists. Therefore, the condition for stability is the same as the condition imposed on the frequency response of part b., i.e., the overall system is stable if $|\alpha| < 1$.

3. **A discrete-time sequence $x[n]$ has the Fourier transform $X(e^{j\omega})$ shown below in Fig. 3.**

   a. **If $x[n]$ is downsampled by a factor $M = 3$ to produce the sequence $x_d[n]$, sketch and label (with specific frequencies, amplitudes, etc.) the Fourier transform of the downsampled sequence $X_d(e^{j\omega})$ if $\omega_H = \pi/2$.**

   b. **What is the maximum value of ω_H that will avoid aliasing if $M = 4$?** **(20 pts)**

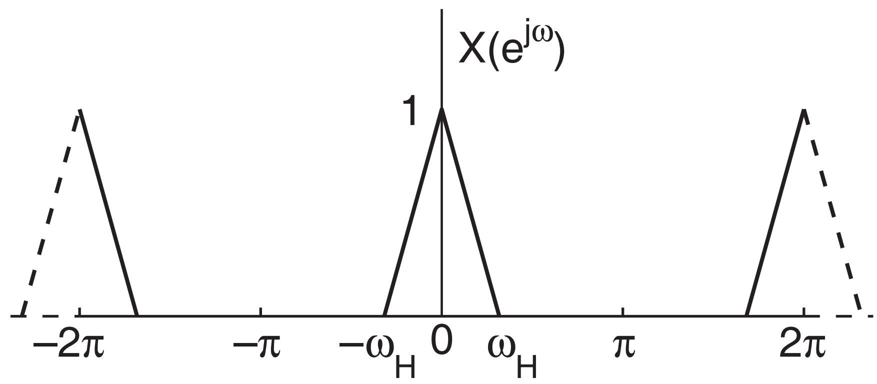
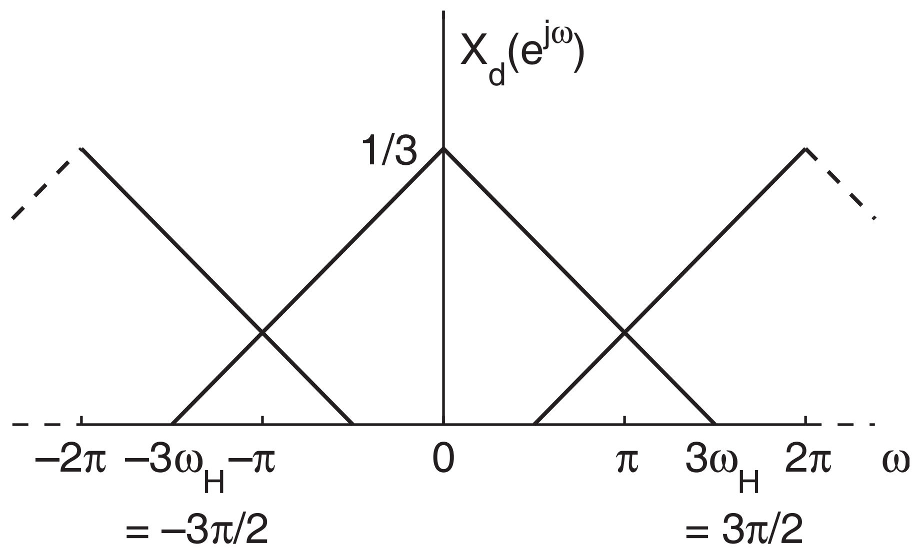


Figure 3: DTFT $X(e^{j\omega})$ for Problem #3.

a. The downsampled spectrum is:



b. There will be no aliasing if the signal is bandlimited to $\pi/M \Rightarrow$ if $M = 4$, the maximum value of ω_H that avoids aliasing is $\pi/4$.

4. **A discrete-time speech waveform originally sampled at 16 kHz is to be added to a music track sampled at 44.1 kHz. The resampling system shown in Fig. 4 below is to be used to upsample the discrete-time speech sequence to 44.1 kHz.**

   a. **What are the minimum values of L and M that can be used, under the constraint that they are both integers?**

   b. **What gain and cutoff frequency are required for the discrete-time lowpass filter?** **(10 pts)**

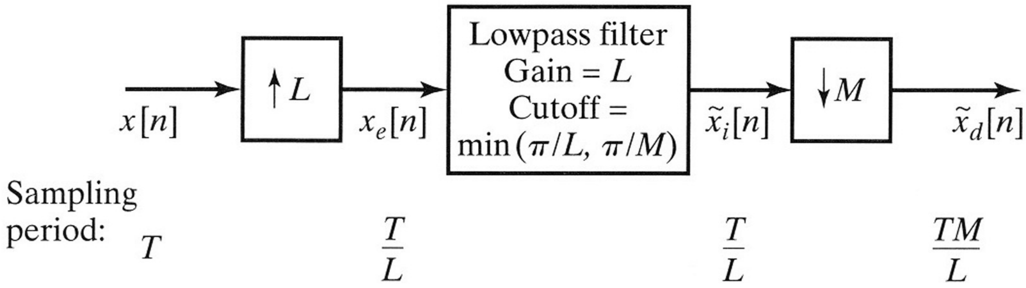


Figure 4: Resampling system for Problem #4.

a. The discrete-time sequence $x[n]$ should be upsampled by the factor $L = 441$ and downsampled by the factor $M = 160$. L and M do not share a common divisor, so these are the lowest integer values of L and M that will resample the 16 kHz signal to a 44.1 kHz signal.

b. The filter gain is L, and the filter cutoff is $\min(\pi/L, \pi/M) = \pi/L = \pi/441 \approx 7.12\times10^{-3}$ radians, since $L > M$.

5. An LTI system has the impulse response $h[n] = 4(-1/3)^n u[n]$. Use the Fourier transform to find the output of this system when the input is $x[n] = (1/2)^n u[n]$. (Note: Tables of Fourier transform properties and Fourier transform pairs are given on the last page.) (20 pts)

From Fourier Transform Pair 4. of Table 2.3, the discrete-time Fourier transforms $H(e^{j\omega})$ and $X(e^{j\omega})$ of the impulse response $h[n]$ and input $x[n]$, respectively, are given by:

$$H\left(e^{j\omega}\right) = \mathcal{F}\{h[n]\} = \frac{4}{1+\frac{1}{3}e^{-j\omega}}, \quad \text{and}$$

$$X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}.$$

From the definition of the transfer function:

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)},$$

the Fourier transform $Y(e^{j\omega})$ of the system output $y[n]$ is:

$$\begin{aligned} Y\left(e^{j\omega}\right) &= H\left(e^{j\omega}\right)X\left(e^{j\omega}\right) \\ &= \frac{4}{1+\frac{1}{3}e^{-j\omega}}\frac{1}{1-\frac{1}{2}e^{-j\omega}} \\ &= \frac{8/5}{1+\frac{1}{3}e^{-j\omega}} + \frac{12/5}{1-\frac{1}{2}e^{-j\omega}}. \end{aligned}$$

From Fourier Transform Pair 4. of Table 2.3, the inverse Fourier transform of $Y(e^{j\omega})$ produces:

$$y[n] = F^{-1}\left\{Y\left(e^{j\omega}\right)\right\} = 8/5(-1/3)^n u[n] + 12/5(1/2)^n u[n].$$