COMP ENG 4TL4:

# Digital Signal Processing

Notes for Lecture #11

Wednesday, October 1, 2003

# 4. THE $z$-TRANSFORM

## 4.1 Definition of the $z$-Transform and the Region of Convergence (ROC)

Consider a continuous-time LTI system with $x(t) = e^{st}$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\; d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\ &= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{=H(s)} = H(s)\, e^{st} \qquad \Rightarrow \end{aligned}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st} dt,\; s = \sigma + j\omega$$ Laplace transform

The Laplace transform is an extremely useful tool for continuous-time LTI system analysis. What about *discrete-time* LTI system analysis?

Let the system input be a discrete-time complex exponential signal:

$$x[n] = z^n \quad \Rightarrow \quad y[n] = H(z)\; z^n \;,$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n} \quad \text{transfer function} \quad \Rightarrow$$

We can introduce:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \quad z\text{−transform}$$

## Relationship between the $z$-transform and the DTFT:

Substitute $z = re^{j\omega}$:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\omega}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left\{x[n]\, r^{-n}\right\} e^{-j\omega n} \\ &= \mathcal{F}\left\{x[n]\, r^{-n}\right\} \qquad \Rightarrow \end{aligned}$$

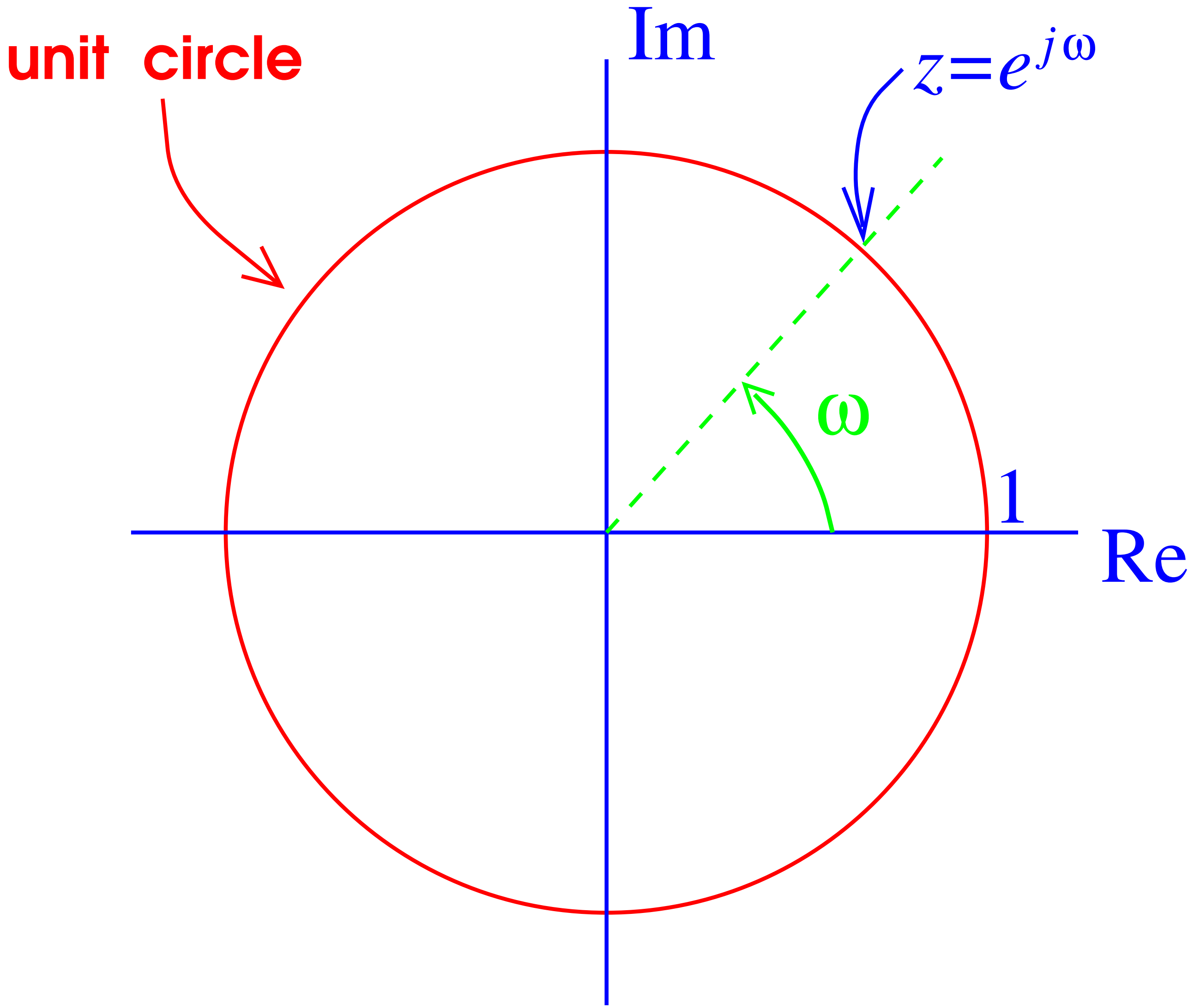
The $z$-transform of an arbitrary sequence $x[n]$ is equivalent to the DTFT of the exponentially-weighted sequence $x[n]r^{-n}$.

If $r = 1$ then:

$$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right) = \mathcal{F}\left\{x[n]\right\} \qquad \Rightarrow$$

the DTFT corresponds to the particular case of the $z$-transform with $|z| = 1$!

The $z$-transform reduces to the DTFT for values of $z$ on the unit circle of the complex $z$-plane:



Question: When does the $z$-transform converge?

Even in the case of finite-energy signals, the $z$-transform does not converge for all values of $z \Rightarrow$ there is a range of values of $z$ — referred to as the Region Of Convergence (ROC) — for which $|X(z)| < \infty$.

Example #1: the $z$-transform of the signal $x[n] = a^n \, u[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] \, z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

For convergence, we require that:

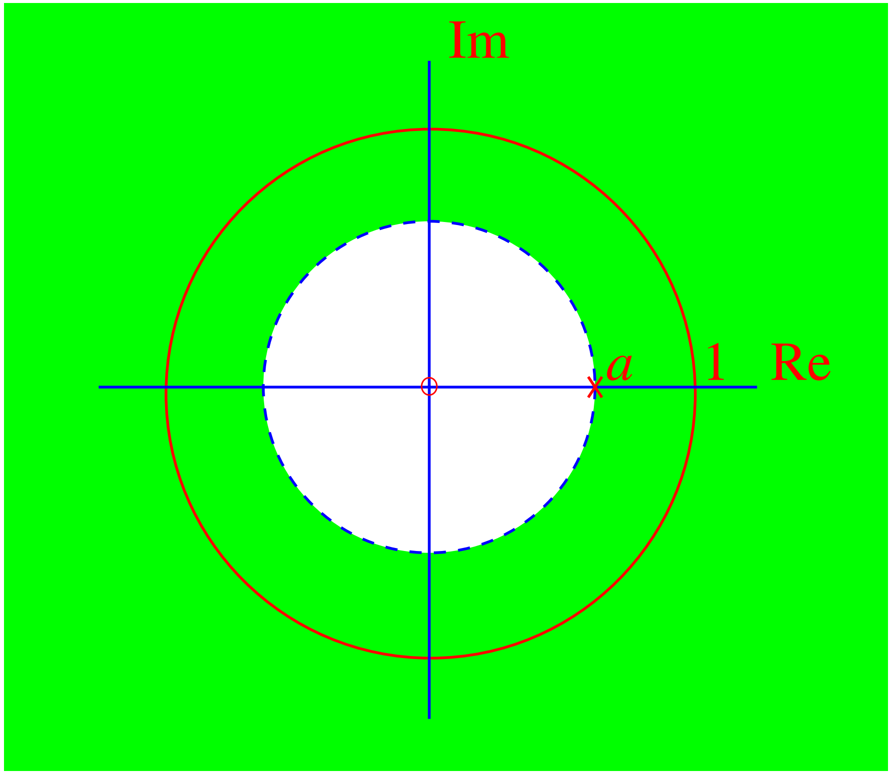
$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

Recall that:

$$\sum_{n=0}^{\infty} |c|^n = \begin{cases} \frac{1}{1-|c|}, & |c| < 1 \\ \infty, & |c| \geq 1 \end{cases} \quad \Rightarrow$$

The ROC is determined by $|z| > |a|$.

Within the ROC:




$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}.$$

Example #2: now, let the signal be $x[n] = -a^n\, u[-n-1]$:

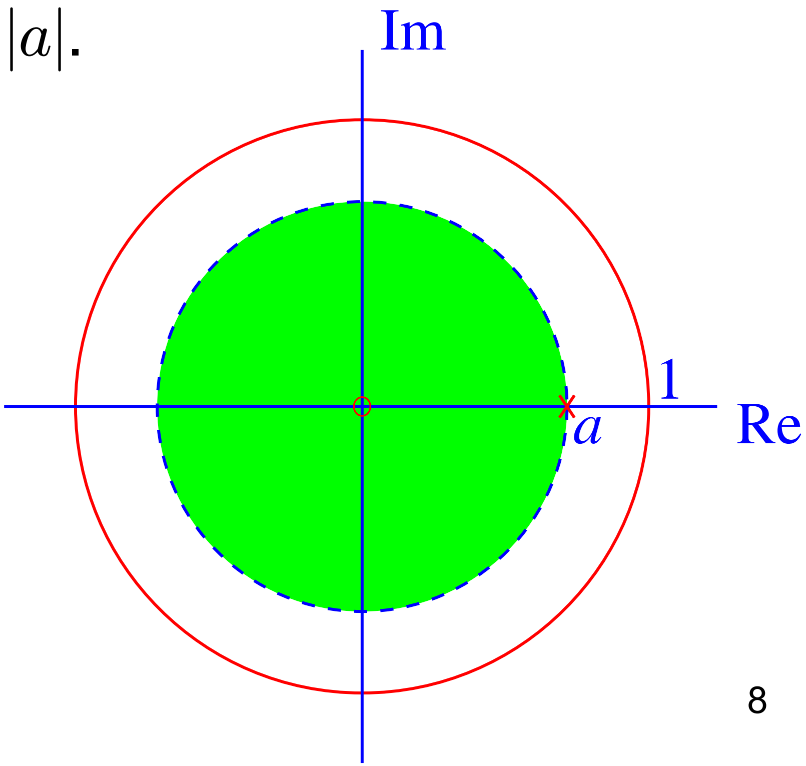
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1]\, z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n \quad \Rightarrow$$

The ROC is determined by $|z| < |a|$.

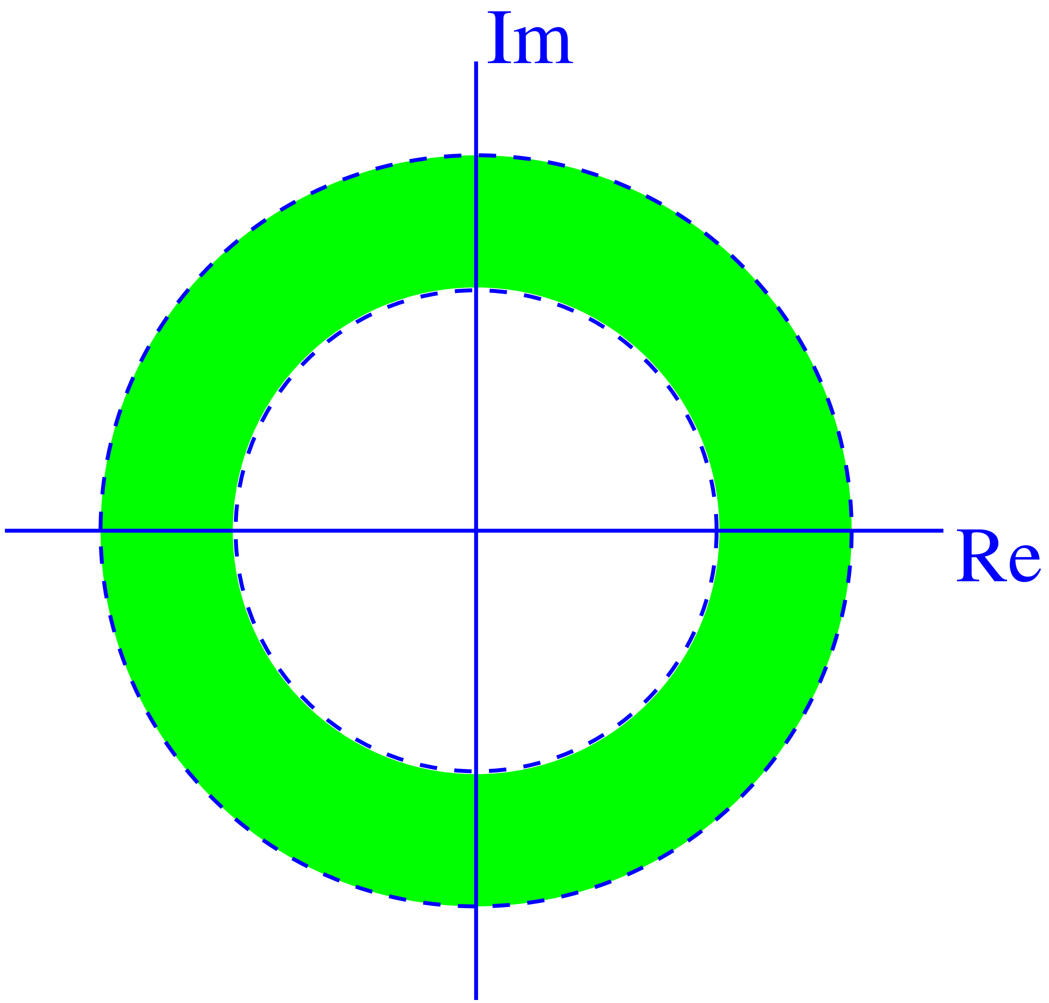
Within the ROC:

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}.$$



# 4.2 Properties of the ROC

Property 1: The ROC of $X(z)$ consists of a ring (or disk) in the $z$-plane centered about the origin.



Property 2: The Fourier transform of $x[n]$ converges absolutely *iff* the ROC of the $z$-transform of $x[n]$ includes the unit circle.

Property 3: The ROC cannot contain any poles.

Property 4: If $x[n]$ is a *finite-duration* sequence, then the ROC is the entire $z$-plane, except possibly $z = 0$ or $z = \infty$.

$$X(z) = \sum_{n=N_1}^{N_2} x[n]\, z^{-n} \quad \text{finite duration signal}$$

Particular cases:

- if $N_1 < 0$ and $N_2 > 0$ then the ROC does not include $z = 0$ and $z = \infty$
- if $N_1 \geq 0$ then the ROC includes $z = \infty$, but does not include $z = 0$
- if $N_2 \leq 0$ then the ROC includes $z = 0$, but does not include $z = \infty$

<u>Property 5:</u> If $x[n]$ is a *right-sided* sequence, then the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

$$X(z) = \sum_{n=N_1}^{\infty} x[n]\, z^{-n} \quad \text{right-sided sequence}$$

<u>Particular cases:</u>

- if $N_1 < 0$ then the ROC does not include $z = \infty$
- if $N_1 \geq 0$ then the ROC includes $z = \infty$

<u>Property 6:</u> If $x[n]$ is a *left-sided* sequence, then the ROC extends inward from the *innermost* (i.e., smallest magnitude) finite pole in $X(z)$ to (and possibly including) $z = 0$.

$$X(z) = \sum_{n=-\infty}^{N_2} x[n]\, z^{-n} \quad \text{left-sided sequence}$$

<u>Particular cases:</u>

- if $N_2 > 0$ then the ROC does not include $z = 0$
- if $N_2 \leq 0$ then the ROC includes $z = 0$

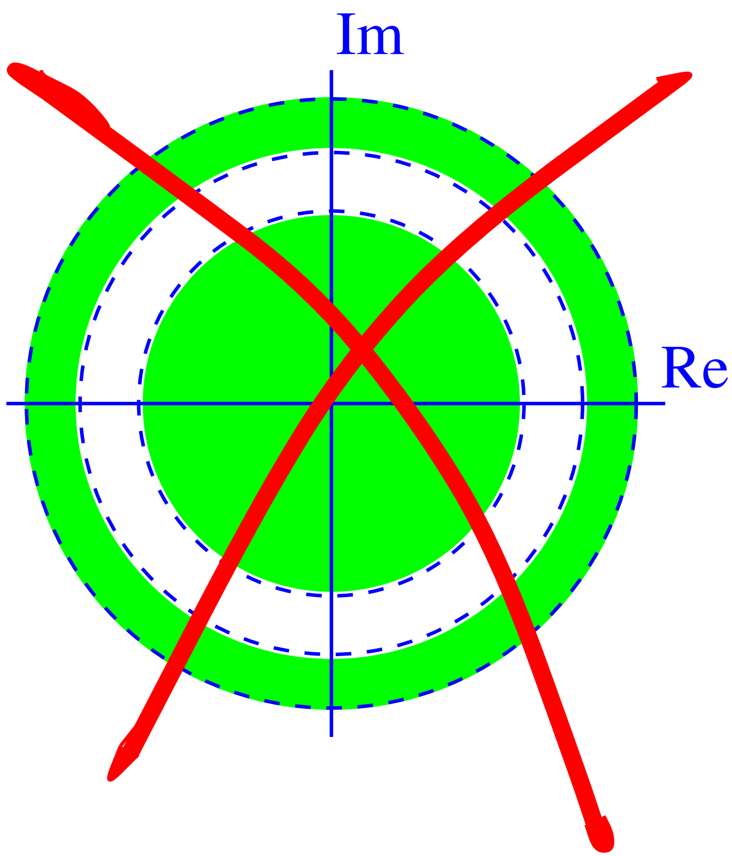
Property 7: If $x[n]$ is a *two-sided* sequence (i.e., neither left-sided nor right-sided), then the ROC consists of a ring in the $z$-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \quad \text{two-sided sequence}$$

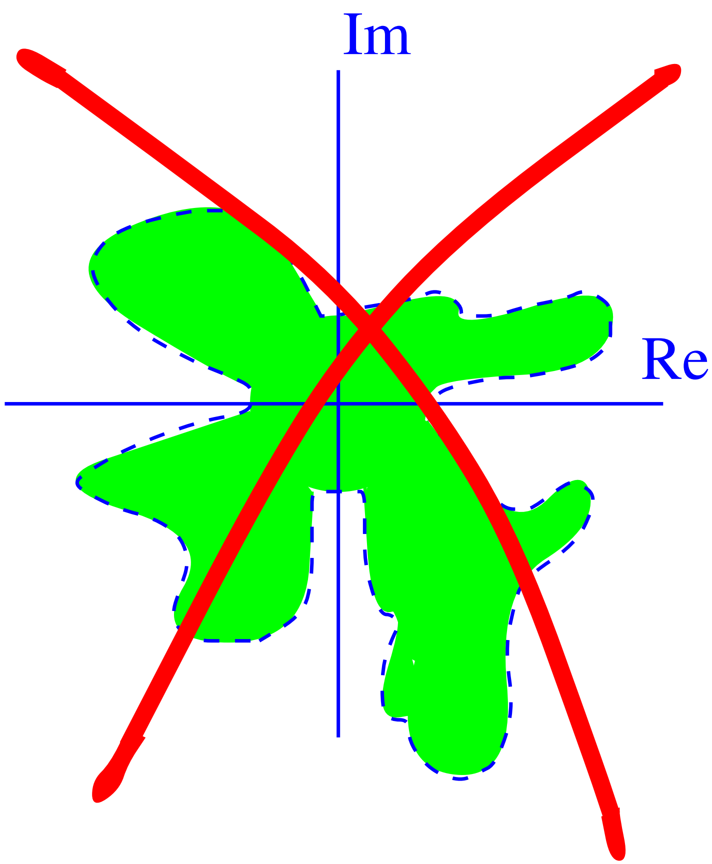
Any two-sided sequence can be represented as a direct sum of a right-sided sequence and a left-sided sequence $\Rightarrow$ the ROC of this composite signal will be the intersection of the ROC's of the components.

Property 8: The ROC must be a connected region.

Re. properties 1 & 8: a) the ROC must be a connected region, and b) the ROC cannot be asymmetric.

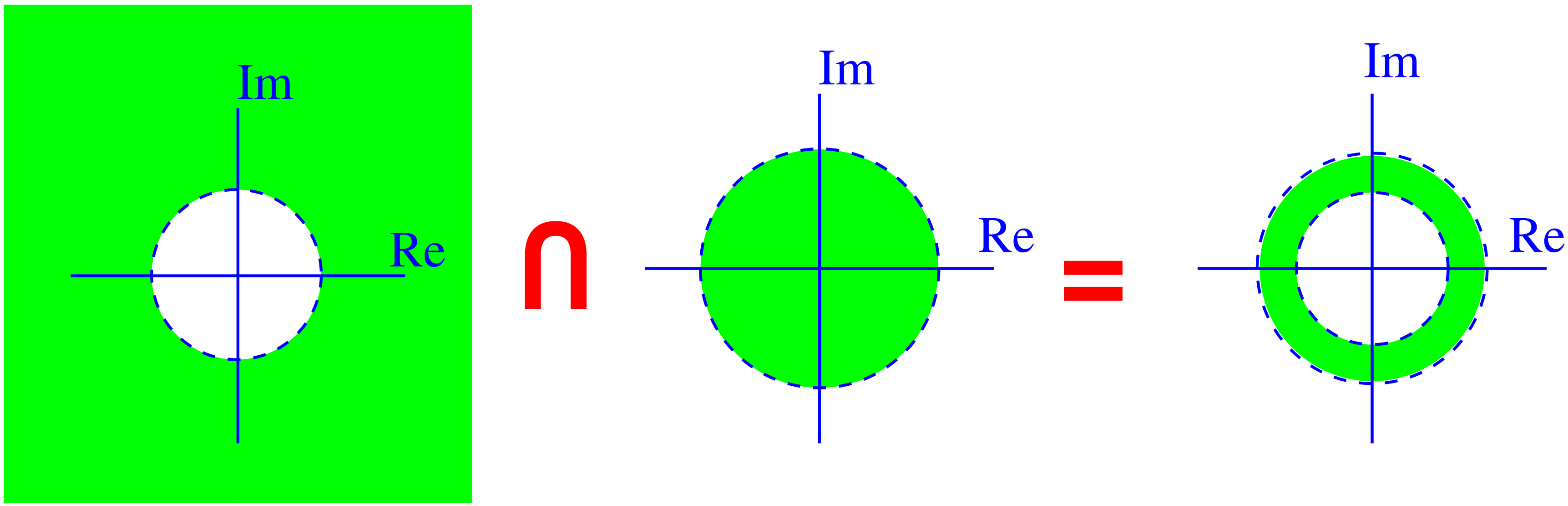



a)



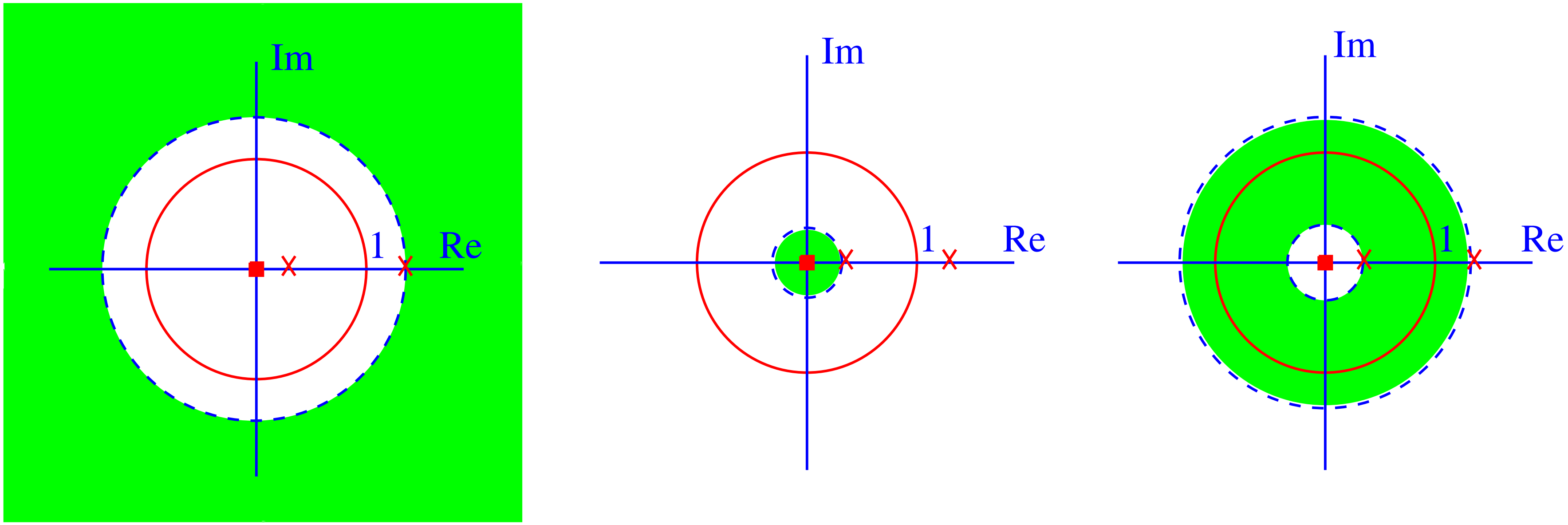

b)

Re. property 7: intersection of the ROC's of right-sided and left-sided sequences $\Rightarrow$ the ROC of a two-sided sequence.



<u>Re. properties 3, 5, 6 & 7:</u> three possible ROC's that correspond to:

$$X(z) = \left[ \left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 1.3z^{-1}\right) \right]^{-1}$$




a) right-sided b) left-sided c) two-sided