COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #13

Tuesday, October 7, 2003

4.6 Analysis of LTI Systems using z-Transforms

From the convolution property:

$$y[n] = h[n] * x[n] \quad \longleftrightarrow \quad Y(z) = H(z)\, X(z)$$

$$H(z) = \mathcal{Z}\{h[n]\} \quad \longleftarrow \quad \text{transfer function}$$

Interpretation of the transfer function:

Properties of LTI systems according to ROCs:

Property 1: A discrete-time LTI system is causal *iff* the ROC of its transfer function $H(z)$ is the exterior of a circle including infinity.

Proof: Follows from properties 4 and 5 of ROCs, since a causal $h[n]$ must be a finite-sequence (FIR) or right-sided sequence (IIR) with $N_1 \geq 0$.

Property 2: A discrete-time LTI system is stable *iff* the ROC of its transfer function $H(z)$ includes the unit circle $|z| = 1$.

Proof: Follows from property 2 of ROCs, since a stable LTI system will have an impulse response that is absolutely summable and consequently the DTFT of the impulse response must exist.

<u>Consequences of properties 1 and 2:</u>

– An FIR linear time-invariant system is inherently stable, because finite-sequences always include the unit circle – see property 4 of ROCs.

– The stability of an IIR linear time-invariant system is dependent on the position of its poles on the z-plane and the "sided-ness" of the impulse response:

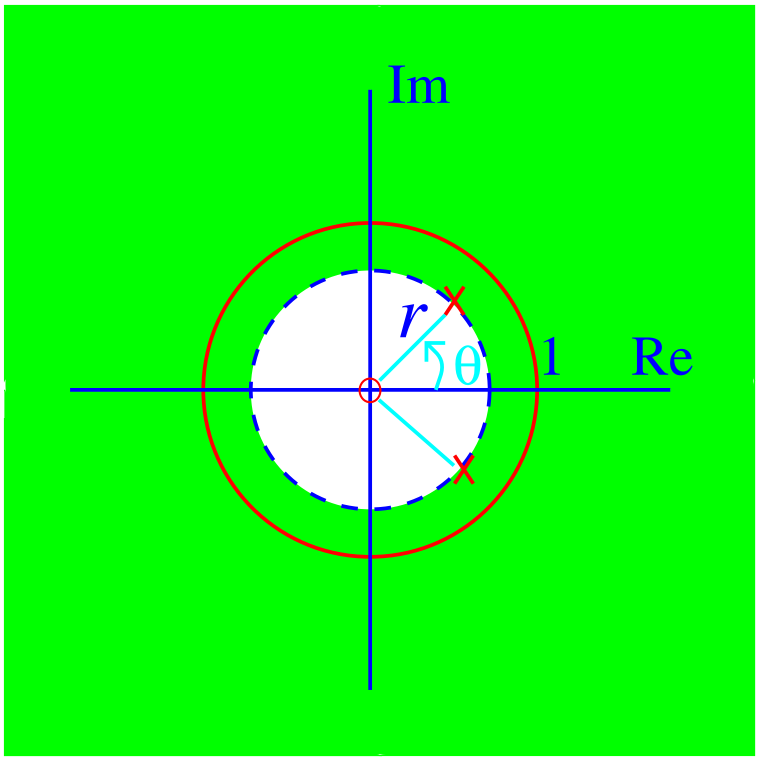
  – If the impulse response is *right-sided* then <u>all the system poles</u> must be <u>inside the unit circle</u> – see property 5 of ROCs.

  – If the impulse response is *left-sided* then <u>all the system poles</u> must be <u>outside the unit circle</u> – see property 6 of ROCs.

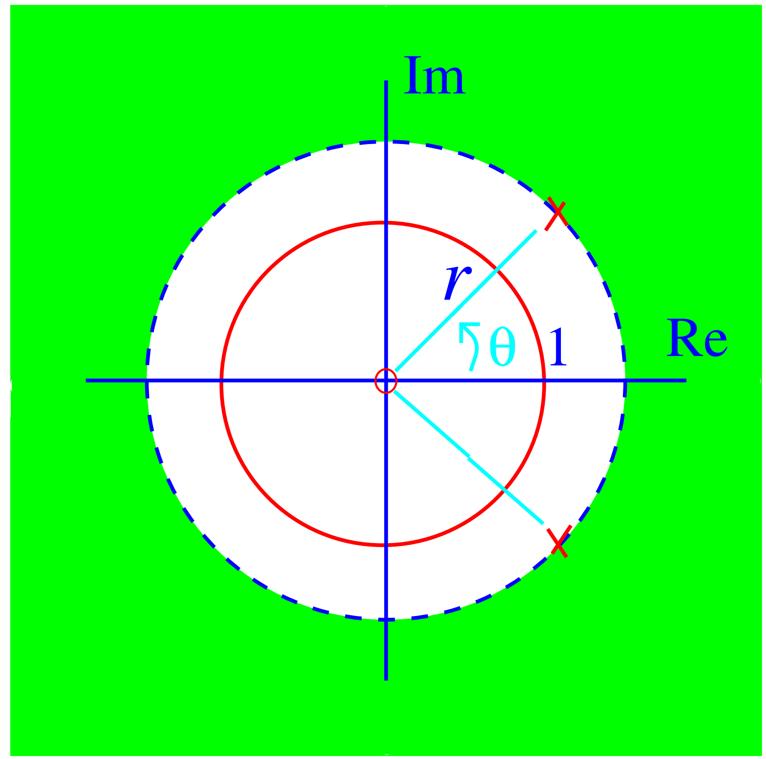
  – If the impulse response is *two-sided* then there must be <u>at least one system pole on each side of the unit circle</u> – see property 7 of ROCs.

Example: IIR, causal LTI system:

$$H(z) = \frac{1}{1 - [2r\cos\theta]z^{-1} + r^2 z^{-2}}$$



stable system ($r<1$)



unstable system ($r>1$)

Application of z-transforms to LCCD equations:

Laplace transforms have the remarkable property of converting continuous-time differential equations to algebraic equations. For example:

$$\mathcal{L}\left\{\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = 3x(t)\right\} \Rightarrow$$
$$s^2Y(s) + 7sY(s) + 12Y(s) = 3X(s)\,,$$

assuming zero initial conditions.

From this equation we can easily obtain the system's transfer function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3}{s^2 + 7s + 12}.$$

Question: The corresponding equations for discrete-time systems are the LCCD equations and the corresponding transform is the z-transform. What does the z-transform of an LCCD equation look like?

Example: $$\mathcal{Z}\{y[n] - 0.5y[n-1] + 0.06y[n-2] = 0.6x[n-1] + 0.3x[n-2]\}.$$

Using the time-shifting property of the z-transform gives:

$$Y(z) - 0.5z^{-1}Y(z) + 0.06z^{-2}Y(z) = 0.6z^{-1}X(z) + 0.3z^{-2}X(z)\,.$$

The LCCD equation has been converted into an algebraic equation, from which we can easily obtain the system's transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.6z^{-1} + 0.3z^{-2}}{1 - 0.5z^{-1} + 0.06z^{-2}}.$$

Recall the LCCD equation for ARMA processes:

$$\sum_{k=0}^{N} a[k]\, y[n-k] = \sum_{k=0}^{M} b[k]\, x[n-k]\,.$$

Taking the z-transforms of both sides of the ARMA equation:

$$\sum_{k=0}^{N} a[k]\, \mathcal{Z}\{y[n-k]\} = \sum_{k=0}^{M} b[k]\, \mathcal{Z}\{x[n-k]\}\,,$$

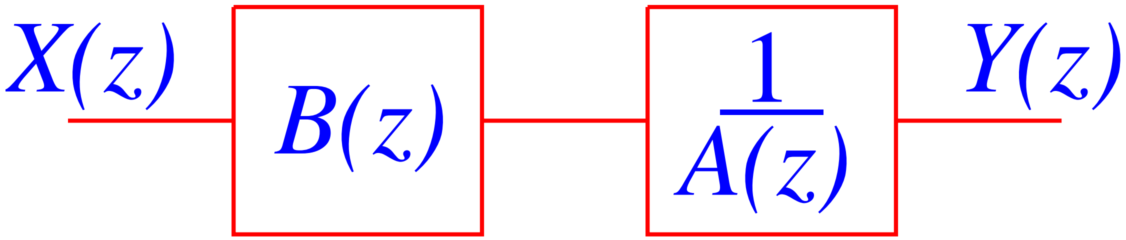
and using the time-shifting property, we obtain:

$$Y(z) \sum_{k=0}^{N} a[k]\, z^{-k} = X(z) \sum_{k=0}^{M} b[k]\, z^{-k}.$$

Hence, the transfer function of an ARMA process is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b[k]\, z^{-k}}{\sum_{k=0}^{N} a[k]\, z^{-k}} = \frac{B(z)}{A(z)}.$$

This equation is referred to as a rational system function. Such systems can be viewed as the cascade of two systems with the transfer functions $B(z)$ and $1/A(z)$, respectively.



The roots of $B(z)$ and $A(z)$ determine the system zeros and poles. An alternative formulation of the transfer function in terms of the zeros and poles is:

$$H(z) = \left(\frac{b[0]}{a[0]}\right) \frac{\prod_{k=1}^{M} \left(1 - c[k]\, z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - d[k]\, z^{-1}\right)}.$$

Note:

- Each factor $(1 - c[k])$ in the numerator contributes an explicit zero at $z = c[k]$ and an implicit pole at $z = 0$.
- Each factor $(1 - d[k])$ in the denominator contributes an explicit pole at $z = d[k]$ and an implicit zero at $z = 0$.
- If $M = N$, then the implicit zeros and poles at $z = 0$ all cancel. If $M \neq N$, then some implicit zeros or poles at $z = 0$ will remain uncancelled.
- If some $a[k]$ or $b[k] = 0$, then some implicit zeros or poles may exist at $z = \infty$.