COMP ENG 4TL4:
Digital Signal Processing

Notes for Lecture #2

Tuesday, September 9, 2003

1.2 Periodic sampling

A discrete-time representation $x[n]$ of a continuous-time signal $x_c(t)$ is obtained through *periodic sampling*:

$$x[n] = x_c(nT), \qquad -\infty < n < \infty,$$

where T is the *sampling period* and its reciprocal, $f_s = 1/T$, is the *sampling frequency*.

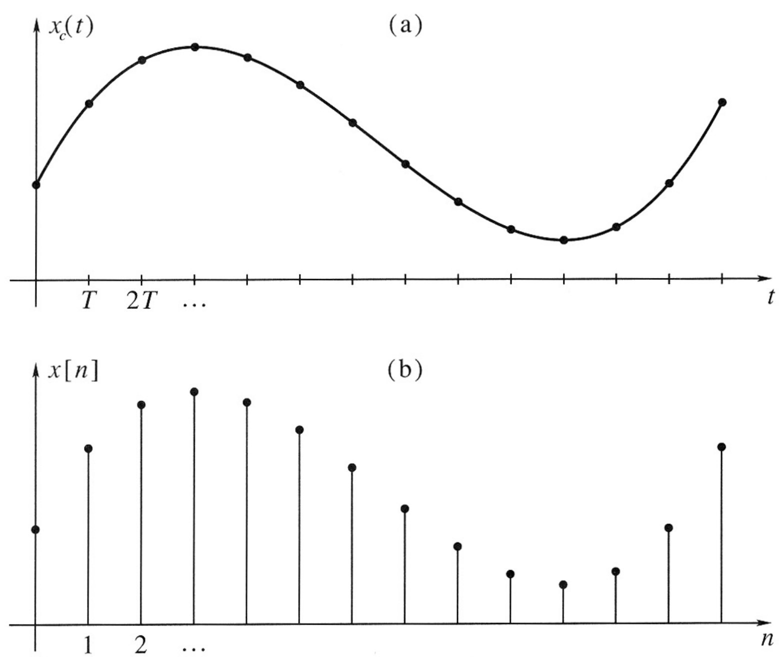
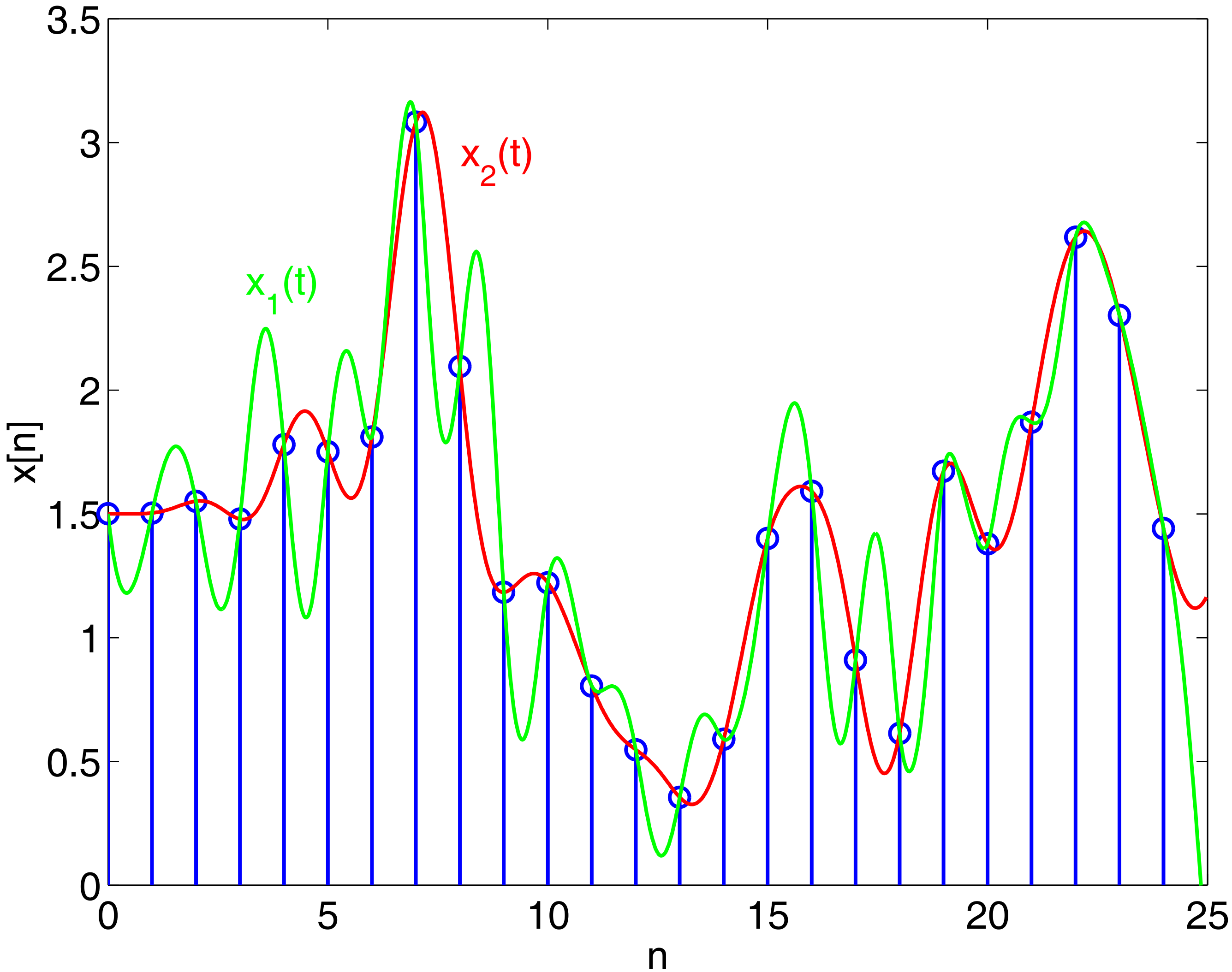


Figure 3.1 The sampling operation: (a) the continuous-time signal $x_c(t)$; (b) the point-sampled sequence $x[n]$.

(Modified from Porat)

Sampling introduces ambiguity into the discrete-time representation, since many continuous-time signals can produce the same sequence of samples.



The ambiguity introduced by sampling can be avoided by restricting the input signals that go into the sampler.

It will be shown that the limiting factor is the relationship between the sampling frequency and the signal *bandwidth* (i.e., the range of frequencies making up a signal). To do this we need to consider the Fourier transform of our discrete-time sequence.

A mathematically simpler way of doing this is to consider the sampling process as two-step procedure, as shown on the next slide, and to calculate the Fourier transform of a continuous-time impulse train representing the sampled signal.

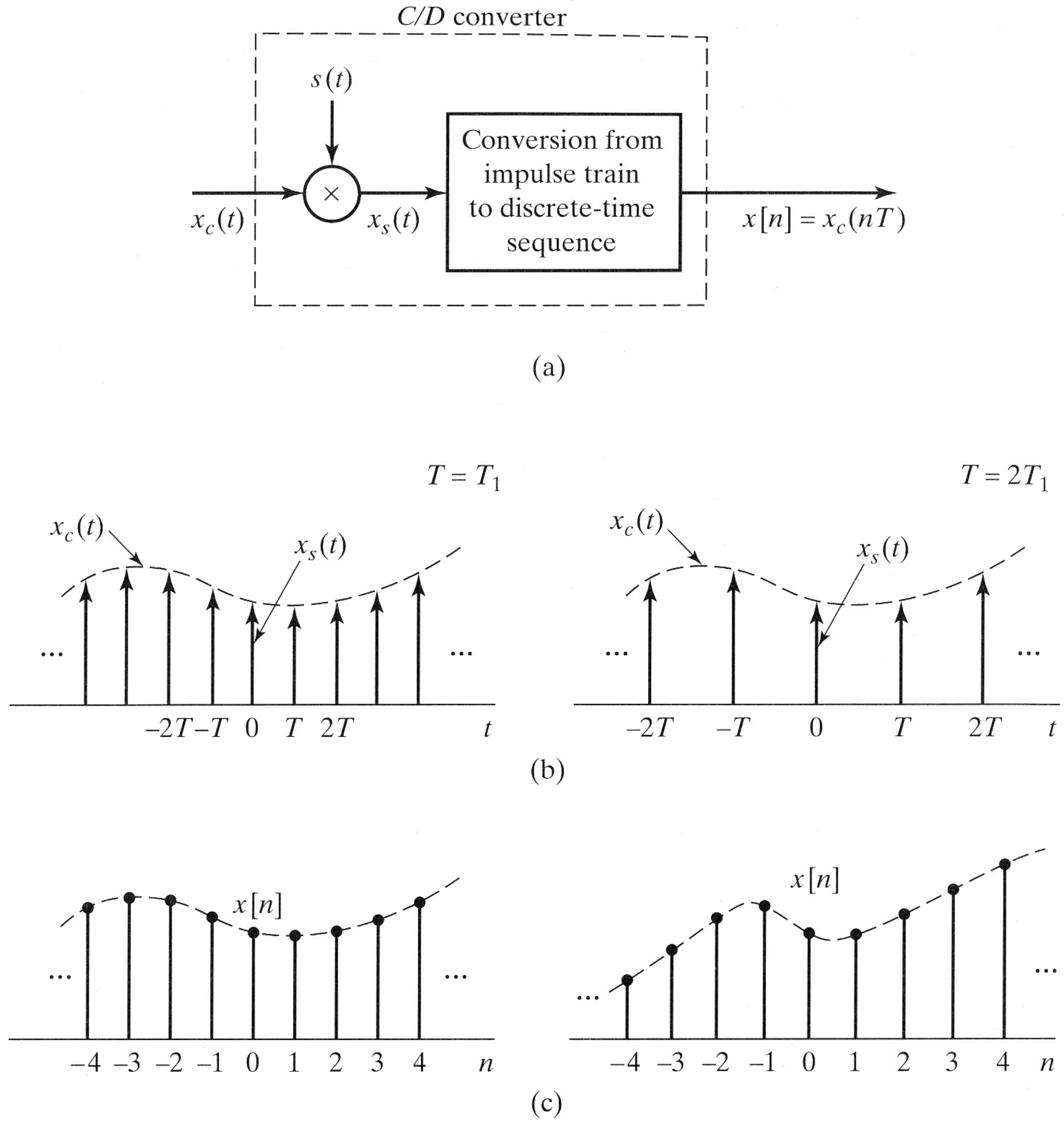


Figure 4.2 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.

(Opppenheim and Schafer)

Periodic sampling can be approximated in continuous-time by multiplying the continuous-time signal with a periodic impulse train:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

where $\delta(t)$ is the unit impulse (Dirac delta) function.

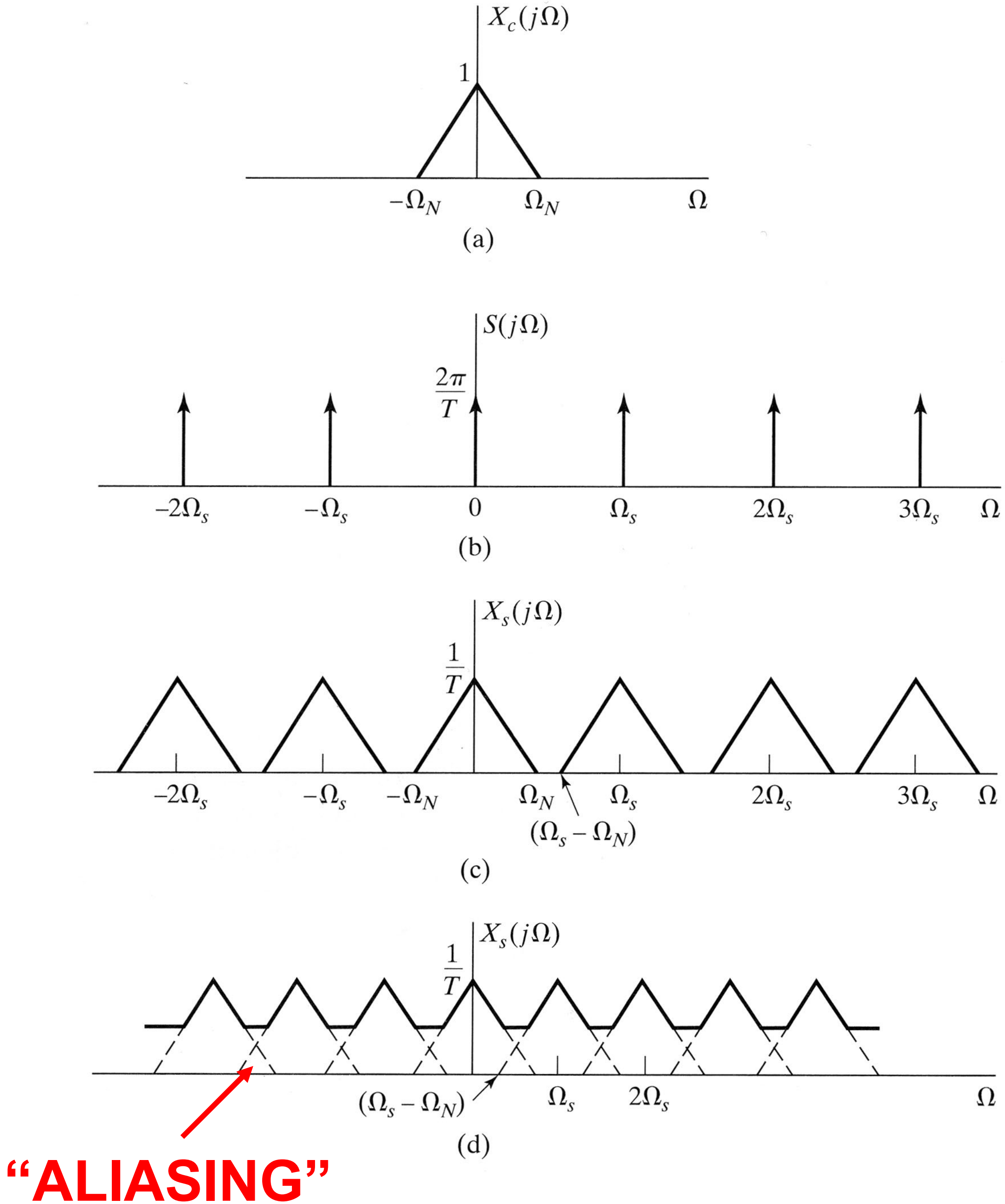
$$\begin{aligned} x_s(t) &= x_c(t)\, s(t) \\ &= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT). \end{aligned}$$

Since $x_s(t)$ is the product of $x_c(t)$ and $s(t)$, the Fourier transform of $x_s(t)$ is the convolution of $X_c(j\Omega)$ and $S(j\Omega)$. The Fourier transform of a periodic impulse train is another periodic impulse train:

$$S\left(j\Omega\right) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\Omega - k\Omega_s\right)$$

where $\Omega_s = 2\pi/T$ is the sampling frequency in radians/s.

$$\begin{aligned} X_s\left(j\Omega\right) &= \frac{1}{2\pi}X_c\left(j\Omega\right) * S\left(j\Omega\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - k\Omega_s\right)\right) \end{aligned}$$



(Opppenheim and Schafer)

Figure 4.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

Nyquist Sampling Theorem:

Let $x_c(t)$ be a *bandlimited* signal with:

$$X_c(j\Omega) = 0 \qquad \text{for } |\Omega| \geq \Omega_N.$$

Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if:

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N.$$

The frequency Ω_N is commonly referred to as the *Nyquist frequency*.

The frequency $2\Omega_N$ that must be exceeded by the sampling rate to avoid aliasing is called the *Nyquist rate*.

To ensure that a signal is bandlimited, an *antialiasing* filter can be used to remove signal components at frequencies above the Nyquist frequency.

1.3 Reconstruction of a bandlimited signal from its samples

According to the sampling theorem, a continuous-time bandlimited signal can be reconstructed from it samples if they are taken frequently enough and the sampling frequency is known.

This is illustrated on the next slide for the continuous-time impulse train representation of the sampled signal.

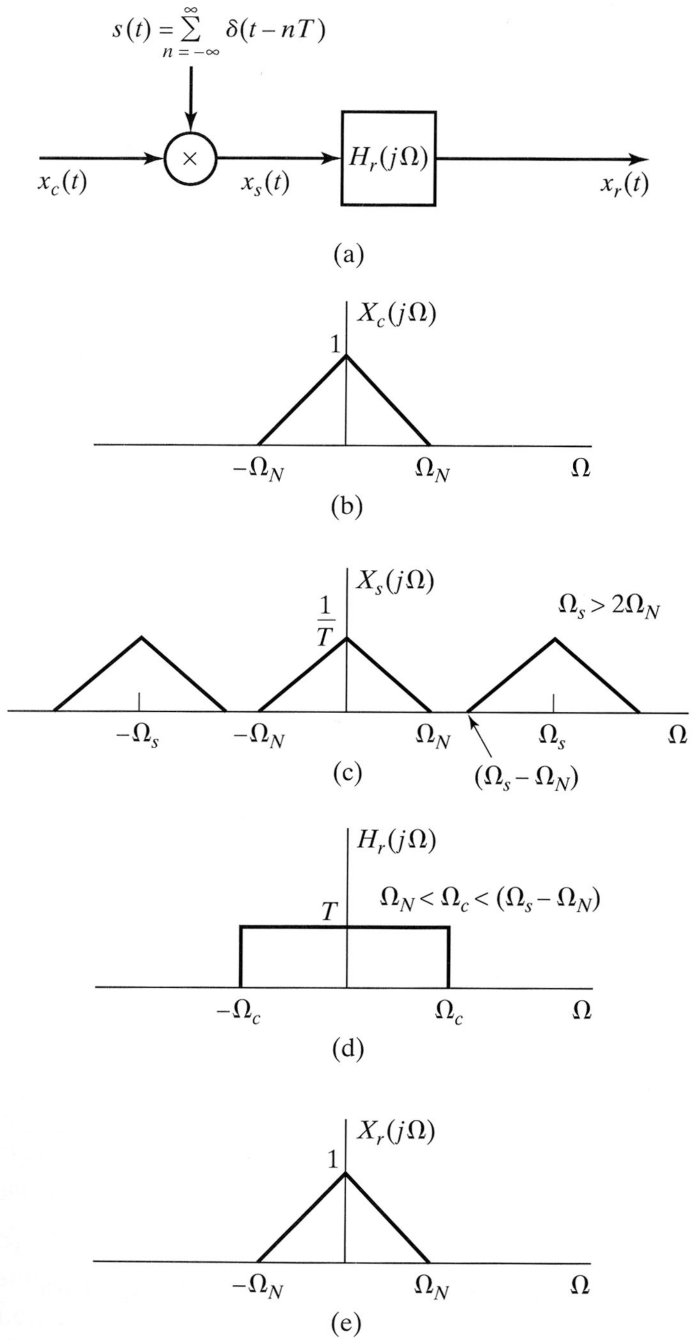


Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

(Opppenheim and Schafer)

It is again convenient to consider the reconstruction process as a two-step procedure and to consider the reconstruction of a continuous-time signal from a continuous-time impulse train representing the sampled signal.

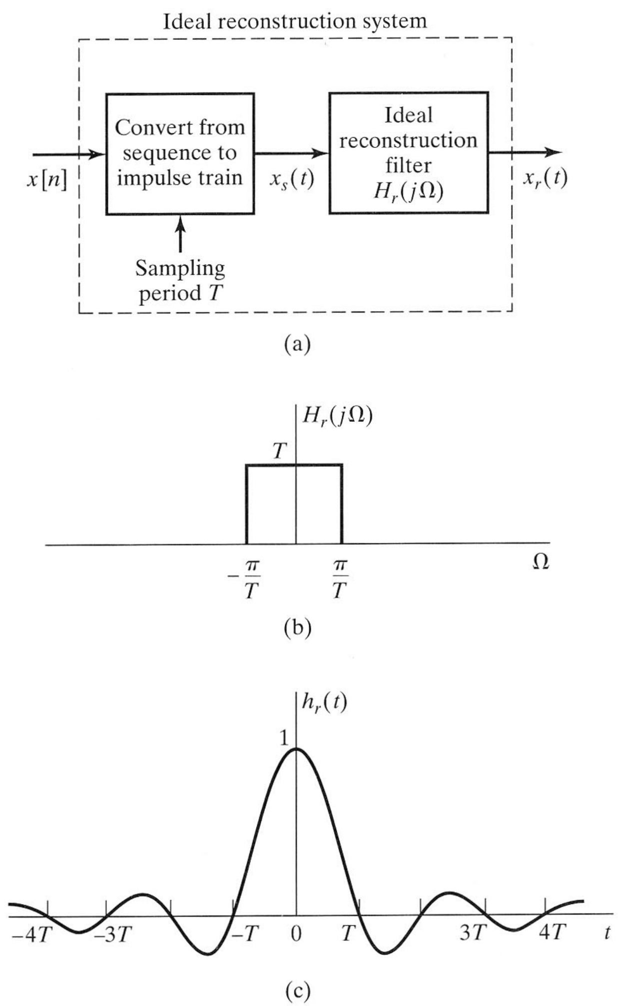


Figure 4.8 (a) Block diagram of an ideal bandlimited signal reconstruction system. (b) Frequency response of an ideal reconstruction filter. (c) Impulse response of an ideal reconstruction filter.

(Opppenheim and Schafer)

Given a sequence of samples $x[n]$, we can form an impulse train:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT).$$

The reconstructed signal $x_r(t)$ is obtained by passing $x_s(t)$ through a reconstruction filter with impulse response $h_r(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t - nT).$$

The ideal lowpass reconstruction filter with cutoff frequency π/T has the impulse response:

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

The ideal reconstruction filter *interpolates* between the sampled values of the continuous-time signal to reconstruct it perfectly, if the Nyquist sampling theorem is satisfied.

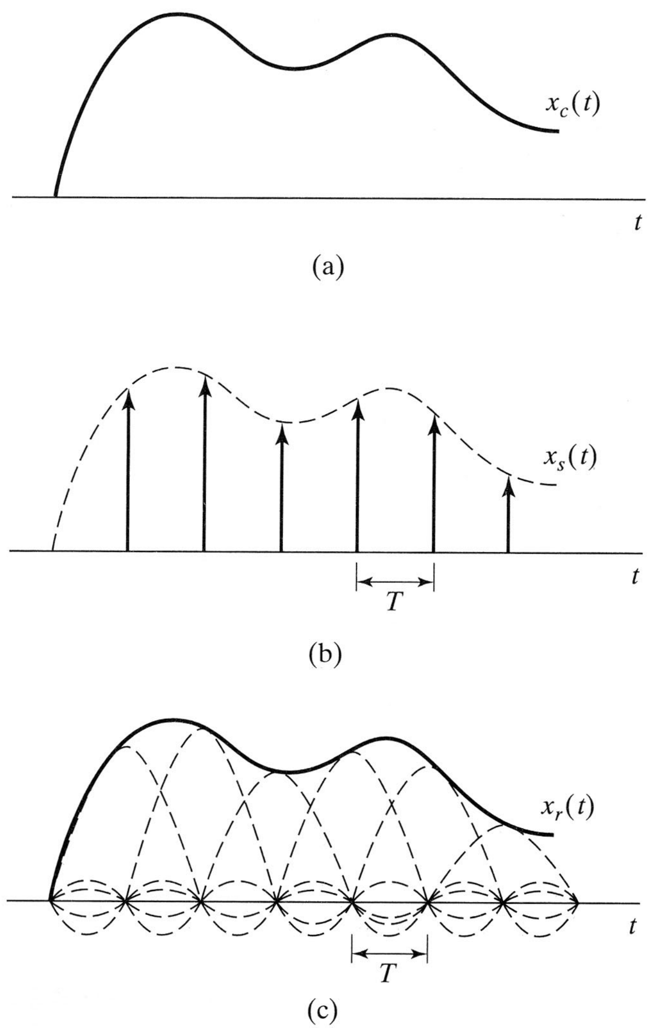


Figure 4.9 Ideal bandlimited interpolation.

(Opppenheim and Schafer)