COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #24

Tuesday, November 4, 2003

6.8 IIR Filter Design

Properties of IIR Filters:

- IIR filters *may be* unstable
- Causal IIR filters with rational system functions have *nonlinear phase*
- An IIR filters of order $\max(M,N)$ can have a sharper frequency response than an FIR filter of order M

Digital IIR Filters from Analog IIR filters:

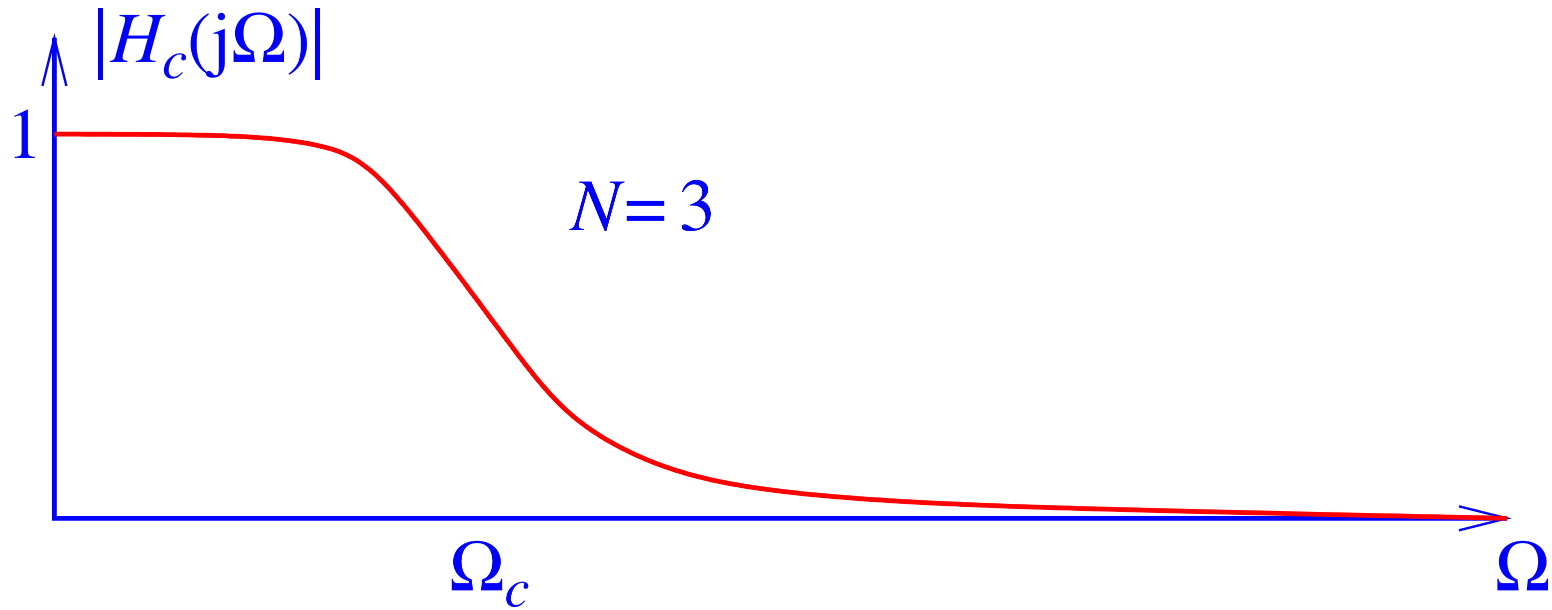
All analog filters are IIR, and the art of continuous-time IIR filter design is very advanced $\Rightarrow$

it makes good sense to design IIR filters in the continuous-time domain and map them to the discrete-time domain

A lowpass Butterworth filter is defined as:

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}},$$

where N is the filter order. Note the monotonic frequency response, i.e., no passband or stopband ripple.



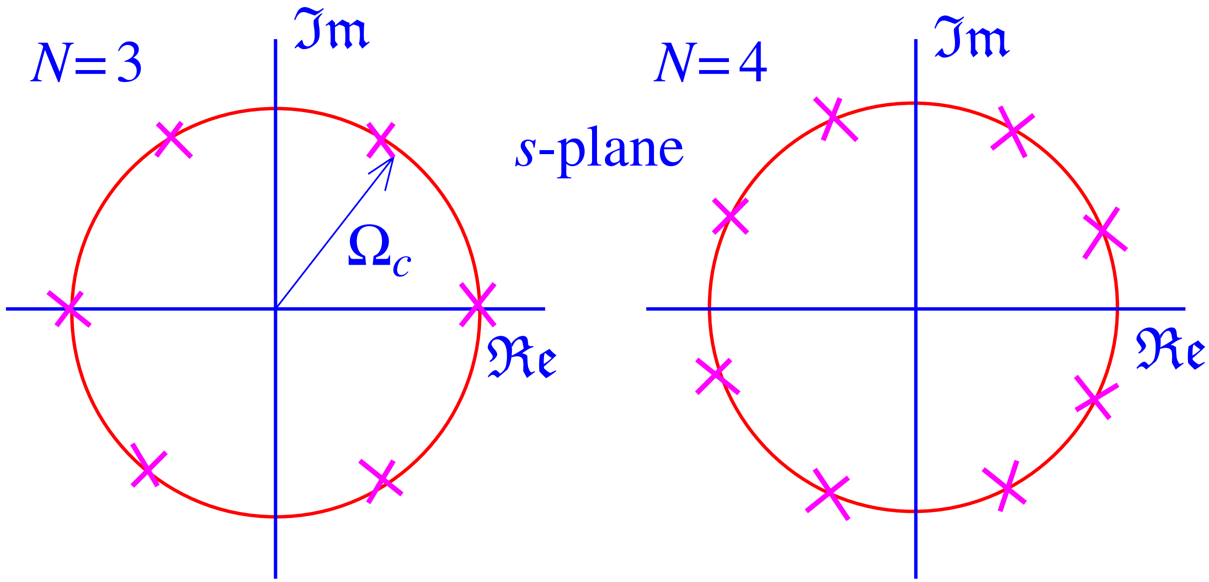
To obtain the transfer function of the Butterworth filter, define $s = j\Omega$ and substitute it into the equation for the magnitude-squared frequency response giving:

$$H_c(s)\,H_c(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}} = \frac{1}{1 + (-1)^N\,(s/\Omega_c)^{2N}}.$$

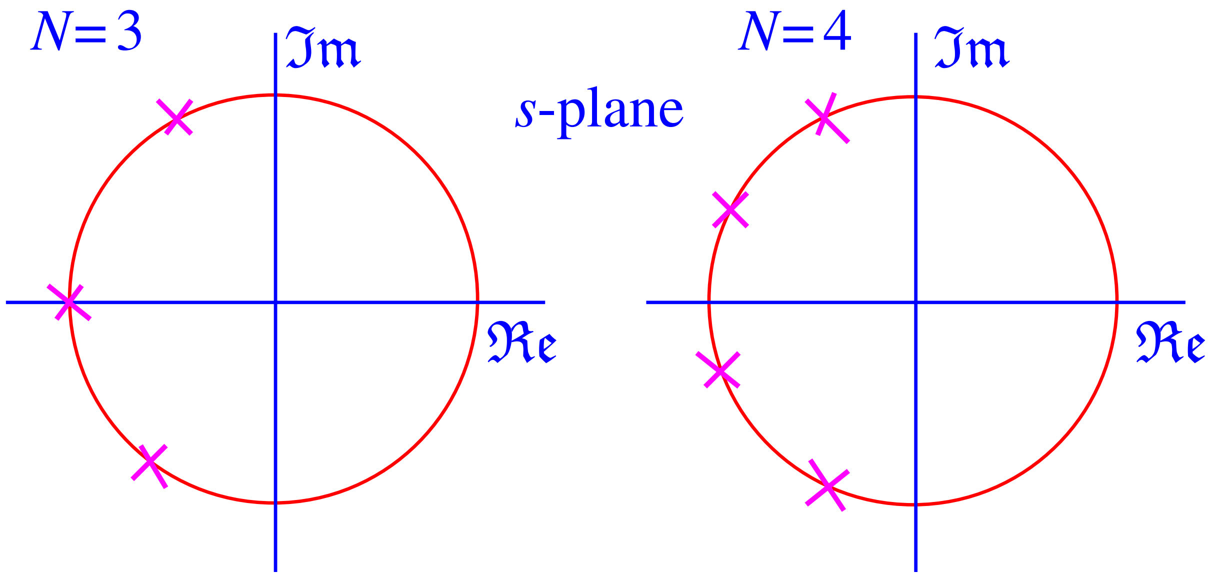
The $2N$ poles of this function are:

$$s_k = \Omega_c \exp\left[j\frac{(N+1+2k)\pi}{2N}\right], \quad 0 \le k \le 2N-1.$$

POLES OF $\left|H_c(s)\right|^2$



POLES OF $H_c(s)$



The Butterworth transfer function can be expressed as:

$$H_c(s) = \prod_{k=0}^{N-1} \frac{-s_k}{s - s_k}, \qquad s = j\Omega,$$

$$s_k = \Omega_c \exp\left[j \left(\frac{\pi}{2} + \frac{(2k+1)\pi}{2N} \right) \right], \qquad 0 \leq k \leq N-1.$$

It is possible to determine Ω_c and N which correspond to $|H_c(j\Omega)|^2$ satisfying some required specifications.

Example: If $N = 1$, then $s_0 = \Omega_c e^{j\pi} = -\Omega_c$. Therefore:

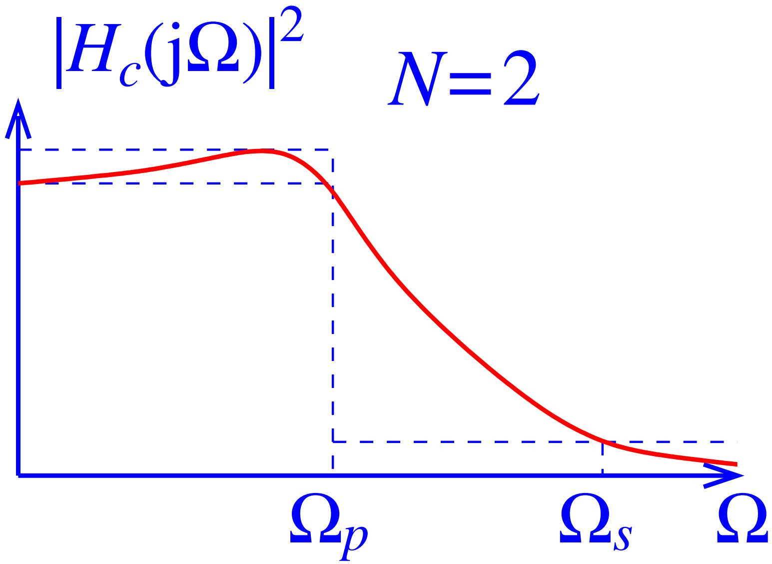
$$H_c(s) = \frac{1}{1 + s/\Omega_c} \quad \Rightarrow \quad |H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^2}.$$

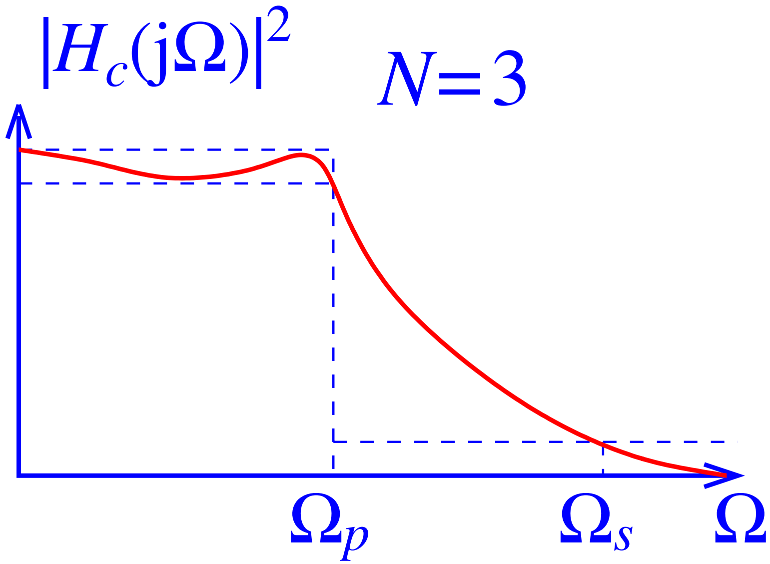
A lowpass Chebyshev Type I filter is defined as:

$$|H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)},$$

where the Chebyshev polynomial of degree N is defined as:

$$T_N(x) = \begin{cases} \cos\left(N \ \text{arccos}\ x\right), & |x| \le 1, \\ \cosh\left(N\, \text{arccosh}\, x\right), & |x| > 1. \end{cases}$$





The s-plane poles and the transfer function of the Chebyshev Type I filter can be obtained in a similar way to that for the Butterworth filter.

Note that a sharper transition region than the Butterworth filter is obtained at the expense of an equiripple passband.

Other widely used analog filters are:-

- Chebyshev Type II: monotonic passband, equiripple stopband; shallower transition region than Chebyshev Type I
- Elliptic: equiripple passband and stopband; sharper transition region than Chebyshev or Butterworth
- Bessel: monotonic frequency response; shallowest transition region; passband group delay nearly constant—note that digital Bessel filters do not retain this property!

Transforming analog filters to digital filters:

An important question is what kind of transformation can be used to obtain digital IIR filters from the corresponding analog filters?

We wish to find some transformations of $H_c(s)$ to $H(z)$, where $H(z)$ is a digital filter transfer function.

Two widely used techniques are:

1. the *impulse invariance* method; and
2. the *bilinear transformation*.

Impulse invariance method of digital filter design:

Consider a (nearly) bandlimited continuous-time filter with the frequency response $H_c(j\Omega)$.

The aim of the impulse invariance method is to obtain a discrete-time frequency response $H(e^{j\omega})$ that is related to the continuous-time frequency response by a linear scaling of the frequency axis, i.e., $\omega = \Omega T$, giving:

$$H\left(e^{j\omega}\right) = H_c\left(j\frac{\omega}{T}\right), \quad |\omega| \leq \pi.$$

Recall that if $h[n] = h_c(nT)$, then:

$$H\left(e^{j\omega}\right) = \frac{1}{T} H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| \leq \pi.$$

The discrete-time frequency response is scaled by a factor of $1/T$ in this case, so we need to scale the impulse response by a factor of T in order to obtain our desired frequency response $\Rightarrow$

$$h[n] = T\, h_c(nT) \qquad \leftarrow \quad \text{impulse invariance transformation}$$

The z-transform of this scaled and sampled impulse response can be taken to determine the discrete-time transfer function of the digital filter. However, it is simpler to carry out the transformation from the s-domain to the z-domain by a direction transformation of a filter's transfer function.

Consider the transfer function of a continuous-time filter expressed in terms of a partial-fraction expansion:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}.$$

(For the sake of simplicity we are only considering the case of single poles at each s_k.)

The corresponding impulse response is:

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

The scaled and sampled impulse response is:

$$h[n] = T\,h_c(nT) = \sum_{k=1}^{N} T\,A_k e^{s_k nT} u[n]$$

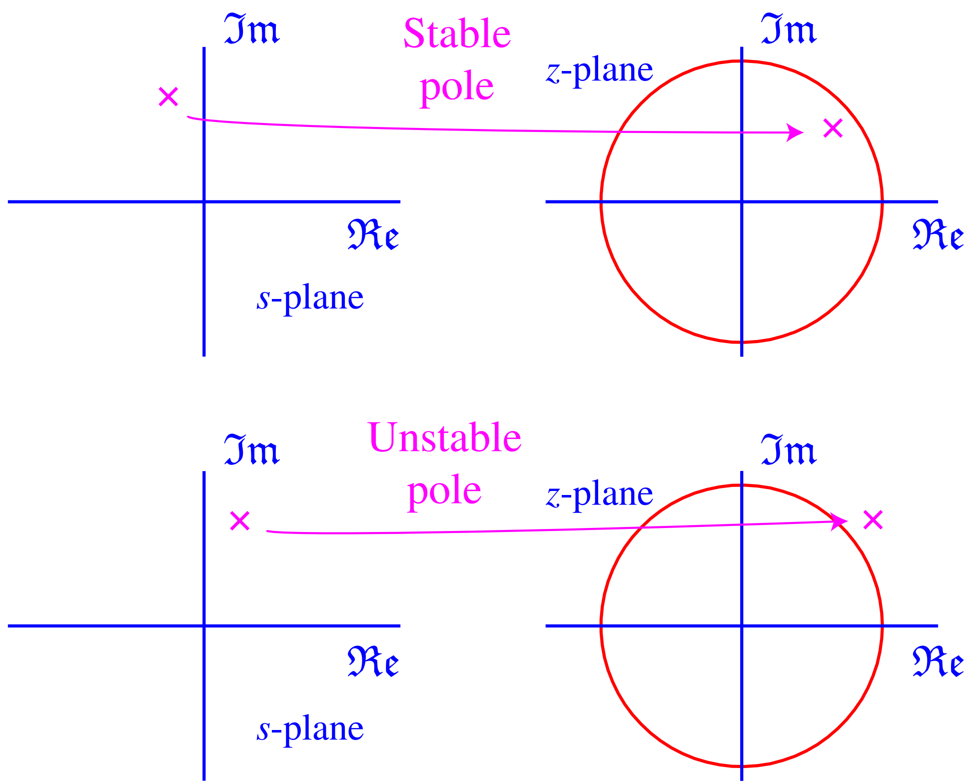
$$= \sum_{k=1}^{N} T\,A_k \left(e^{s_k T}\right)^n u[n]\,.$$

The system function of the discrete-time filter is therefore given by:

$$H(z) = \sum_{k=1}^{N} \frac{T\,A_k}{1 - e^{s_k T} z^{-1}}.$$

Note that a pole at $s = s_k$ on the s-plane is mapped to a pole at $z = e^{s_k T}$ on the z-plane.

Therefore, a stable pole in the s-domain ($\mathfrak{Re}\{s_k\} < 0$) will map to a stable pole in the z-domain ($|z_k| = e^{\mathfrak{Re}\{s_k T\}} < 1$).



However, the zeros do not map in such a simple way. The discrete-time zeros will be a function of the discrete-time poles and the coefficients TA_k in the partial fraction expansion.