COMP ENG 4TL4:

Digital Signal Processing

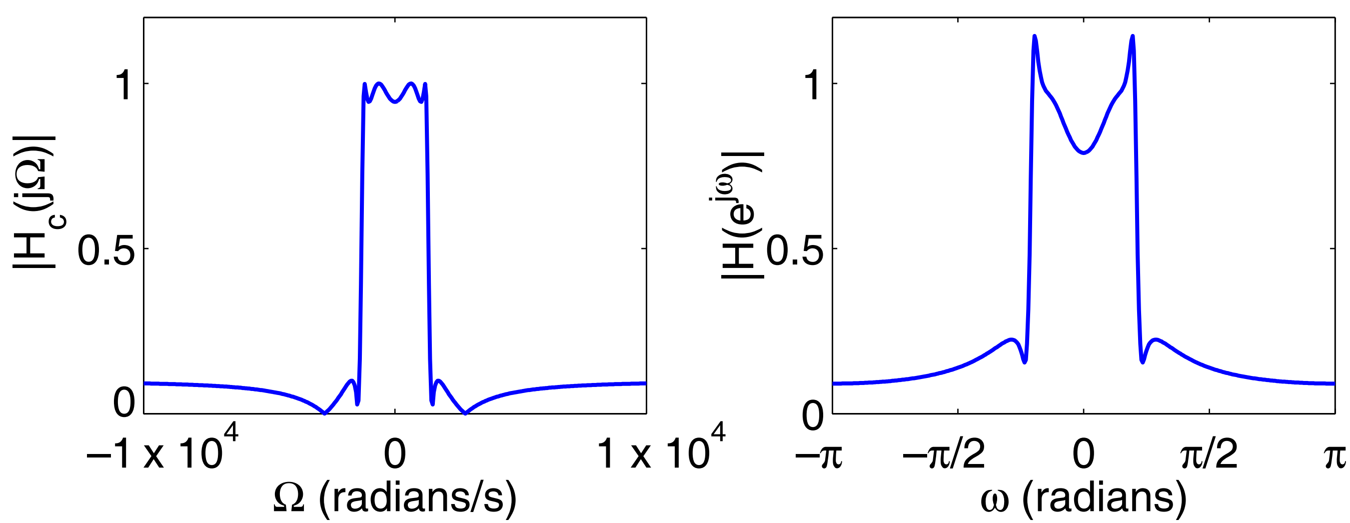
Notes for Lecture #25

Wednesday, November 5, 2003

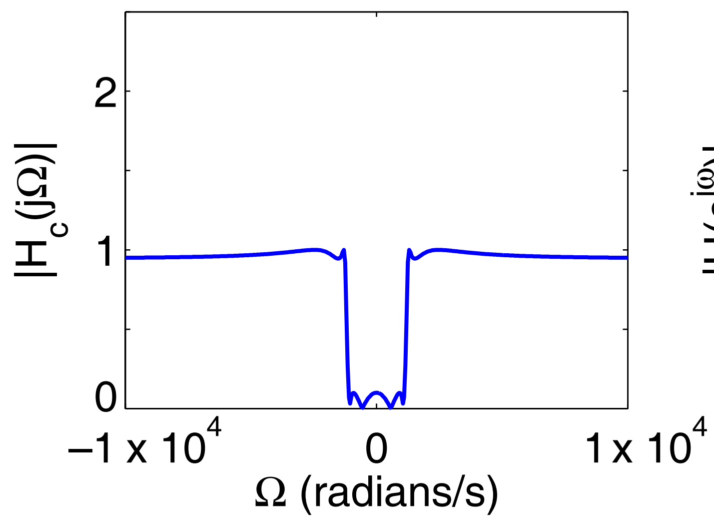
Aliasing in the impulse invariance method:

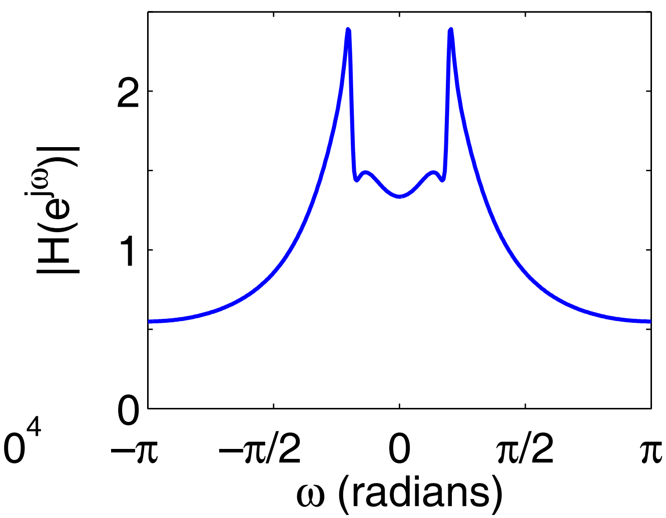
The impulse invariance method is only suitable for filters with a *bandlimited frequency response*; highpass or bandstop analog filters would require additional bandlimiting to avoid severe aliasing.

Consider the case of an elliptic lowpass filter of order $N = 4$ and cutoff frequency $\Omega_c = 2\pi\times 200 \ \mathrm{radian/s}$ that is transformed via the impulse invariance method with a sampling frequency $f_s = 2 \ \mathrm{kHz}$:



Now consider the case of an elliptic highpass filter of order $N = 4$ and cutoff frequency $\Omega_c = 2\pi{\times}200\ \mathrm{radian/s}$ that is transformed via the impulse invariance method with a sampling frequency $f_s = 2\ \mathrm{kHz}$:





Note that the aliasing is so severe that the continuous-time *highpass* filter is mapped to a discrete-time *lowpass/bandpass* filter by the impulse invariance method.

Bilinear transformation method of digital filter design:

The aliasing problems with the impulse invariance method motivates the bilinear transformation, in which the entire $j\Omega$-axis in the s-plane is mapped to one revolution of the unit circle in the z-plane via the algebraic transformation:

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$\Rightarrow \qquad H(z) = H_C(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Since $-\infty \leq \Omega \leq \infty$ maps onto $-\pi \leq \omega \leq \pi$, the transformation between the continuous-time and discrete-time frequency variables must be nonlinear.

Solving for z gives:

$$z = \frac{1 + (T/2)s}{1 - (T/2)s},$$

and substituting for $s = \sigma + j\Omega$, we obtain:

$$z = \frac{1 + \sigma T/2 + j\Omega T/2}{1 - \sigma T/2 - j\Omega T/2}.$$

If $\sigma < 0$, then $|z| < 1$ for any value of $\Omega \Rightarrow$

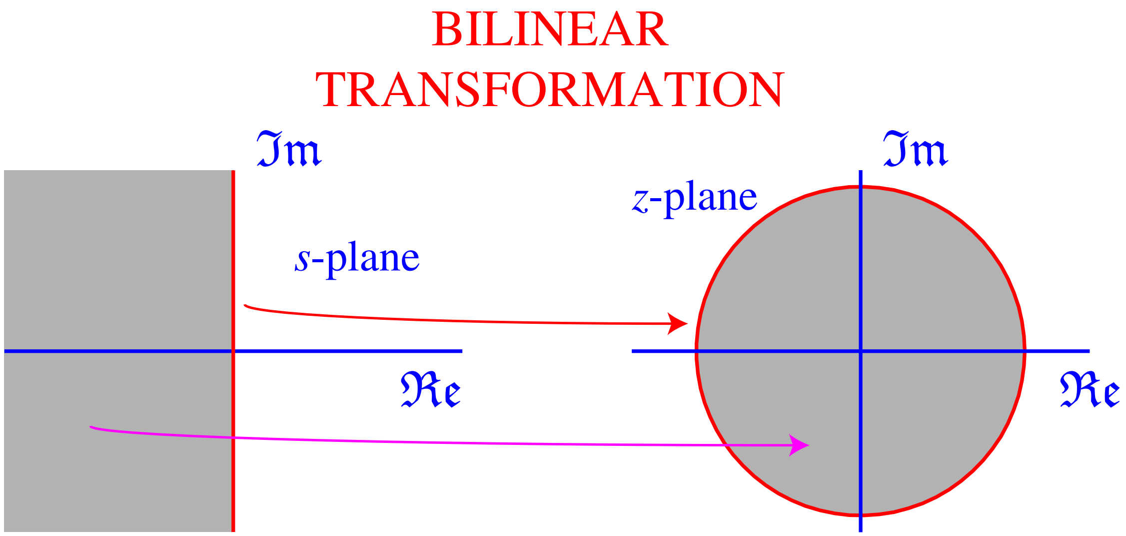
the entire left-half of the s-plane maps to the inside of the unit circle in the z-plane, such that causal stable continuous-time filters map to causal stable discrete-time filters.

Substituting $s = j\Omega$ gives:

$$z = \frac{1 + j\Omega T/2}{1 - j\Omega T/2}.$$

Note that $|z| = 1$ for any value of Ω, so the $j\Omega$-axis maps onto the unit circle, such that:

$$e^{j\omega} = \frac{1 + j\Omega T/2}{1 - j\Omega T/2}.$$



Returning to our algebraic expression for the bilinear transformation and substituting $z = e^{j\omega}$, we obtain:

$$s = \frac{2}{T}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right),$$

or equivalently:

$$\begin{aligned} s &= \sigma + j\Omega = \frac{2}{T}\left[\frac{2e^{-j\omega/2}\left(j\sin\omega/2\right)}{2e^{-j\omega/2}\left(\cos\omega/2\right)}\right] \\ &= \frac{2j}{T}\tan\left(\omega/2\right). \end{aligned}$$

Equating the real and imaginary parts of both side of the equation above gives $\sigma = 0$ and:

$$\Omega = \frac{2}{T}\tan\left(\omega/2\right) \;\leftrightarrow\; \omega = 2\arctan\left(\Omega T/2\right)$$

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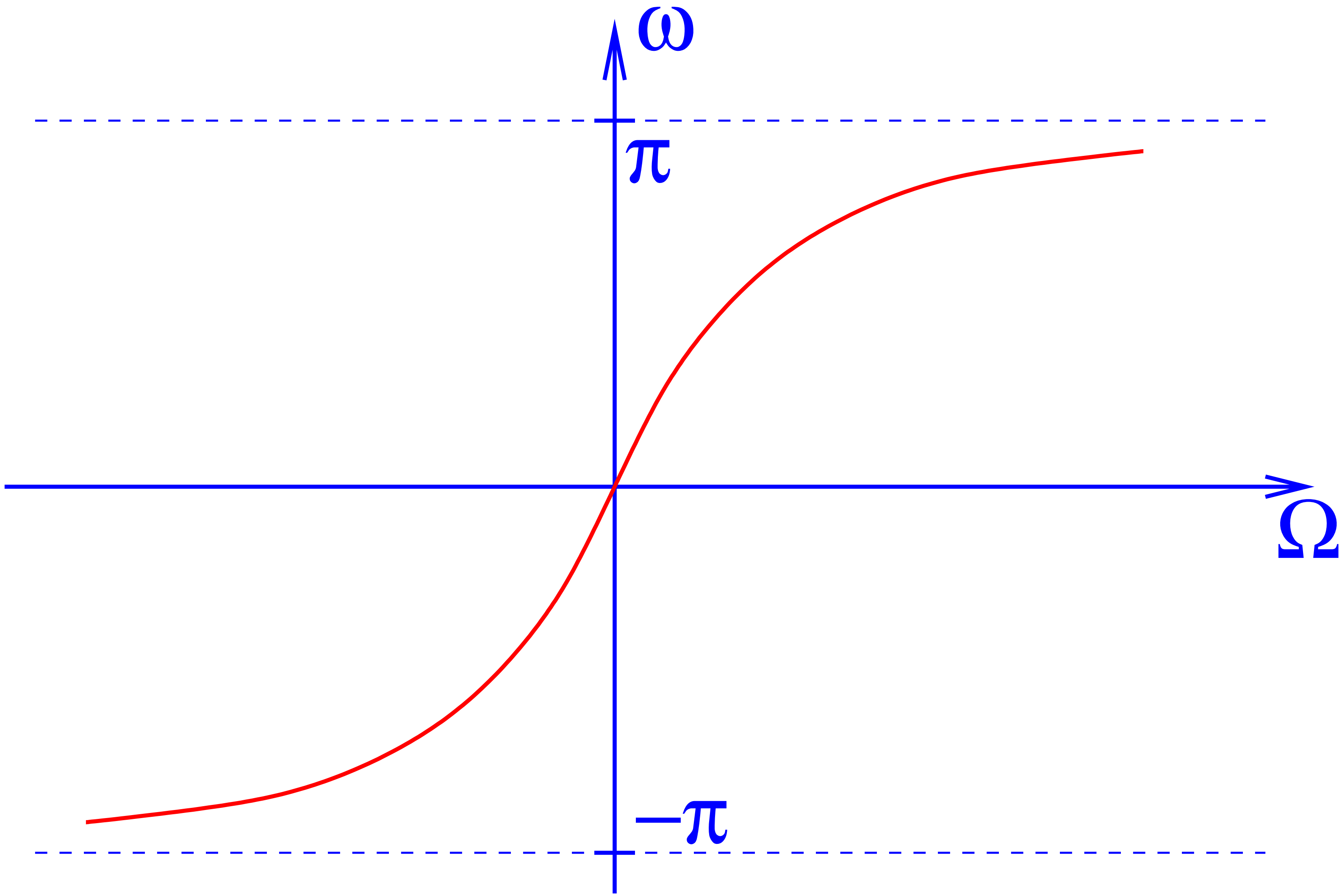


The bilinear transformation *warps* the digital frequency with respect to analog frequency. The *nonlinear warping function* is $2\arctan\{\Omega T/2\}$.

To design a lowpass filter with the digital transition-region frequencies ω_p and ω_s, we find the analog prewarped frequencies:

$$\Omega_p = \frac{2}{T}\tan\left(\omega_p/2\right) \quad \text{and} \quad \Omega_s = \frac{2}{T}\tan\left(\omega_s/2\right)$$

and design the analog filter using these transformed specifications.

Then, the analog filter can be transformed to a digital filter via:

$$H(z) = H_c(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Bilinear transformation IIR filter design method:

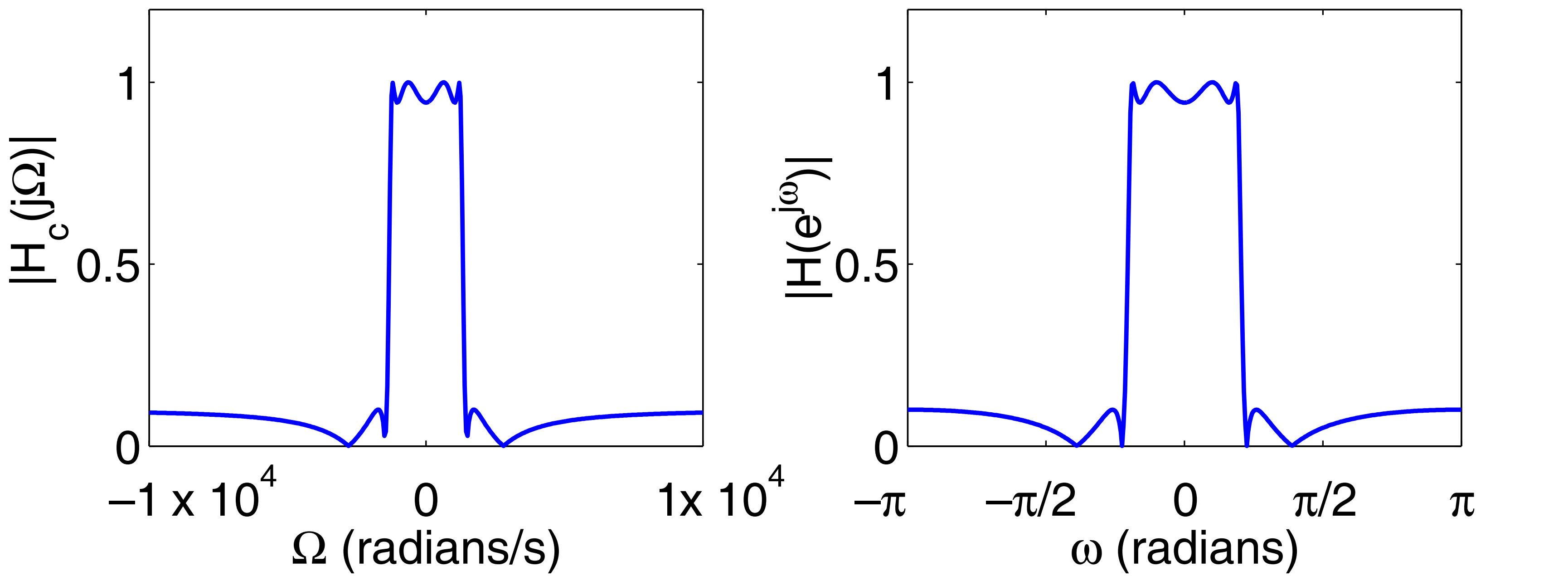
Step 1: Convert each specified edge-band (transition region) frequency of the desired digital filter to a corresponding edge-band frequency of an analog filter

Step 2: Design an analog filter $H(s)$ of the desired type, according to the transformed specifications

Step 3: Transform the analog filter $H(s)$ to a digital filter $H(z)$ using the bilinear transform

Let us reconsider the case of an elliptic lowpass filter of order $N = 4$ and cutoff frequency $\Omega_c = 2\pi{\times}200 \text{ radian/s}$.

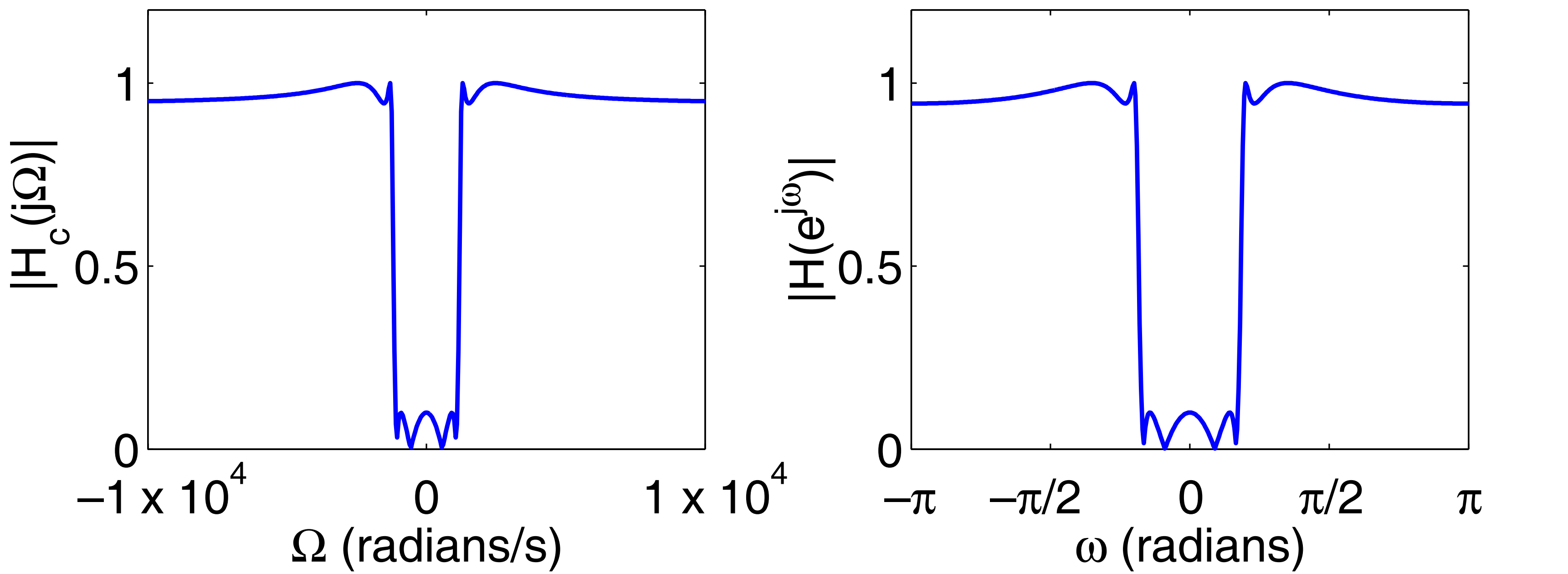
Now let us apply the bilinear transformation with a sampling frequency $f_s = 2 \text{ kHz}$:



Note the absence of aliasing when using the bilinear transformation!

Now let us reconsider the case of an elliptic <u>highpass</u> filter of order $N = 4$ and cutoff frequency $\Omega_c = 2\pi{\times}200$ radian/s.

Let us apply the bilinear transformation with a sampling frequency $f_s = 2$ kHz:



Again note the absence of aliasing when using the bilinear transformation, even for the highpass filter!

Optimal methods for IIR filter design:

Direct design of IIR filters (i.e., not requiring analog filter design) can be achieved by exploiting the ARMA model:

$$\sum_{k=0}^{N} a[k]\, y[n-k] = \sum_{k=0}^{M} b[k]\, x[n-k]\,.$$

This model corresponds to an IIR filter with the frequency response:

$$H\left(e^{j\omega}\right) = \frac{\sum_{k=0}^{M} b[k]\, e^{-j\omega k}}{\sum_{k=0}^{N} a[k]\, e^{-j\omega k}}.$$

The Yule-Walker recursive method of IIR filter design performs a least-squares fit to the ideal magnitude response $|H_{\mathrm{id}}(e^{j\omega})|$ by minimizing the expression:

$$\epsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|H_{\mathrm{id}}\left(e^{j\omega}\right)| - \left|\frac{\sum_{k=0}^{M} b[k]\, e^{-j\omega k}}{\sum_{k=0}^{N} a[k]\, e^{-j\omega k}}\right|\right)^2 d\omega,$$

to obtain the filter parameters $a[k],\ k = 0,\ldots,N$ and $b[k],\ k = 0,\ldots,M$.

- Note that we have set no constraint on the phase response of the filter, so we need to check that the phase response of the resulting filter is acceptable for our particular application.
- Other design methods exists in which we can specify a particular phase response, but since we cannot obtain a causal linear-phase IIR filter with the ARMA model, it is difficult to know what phase response should be specified.