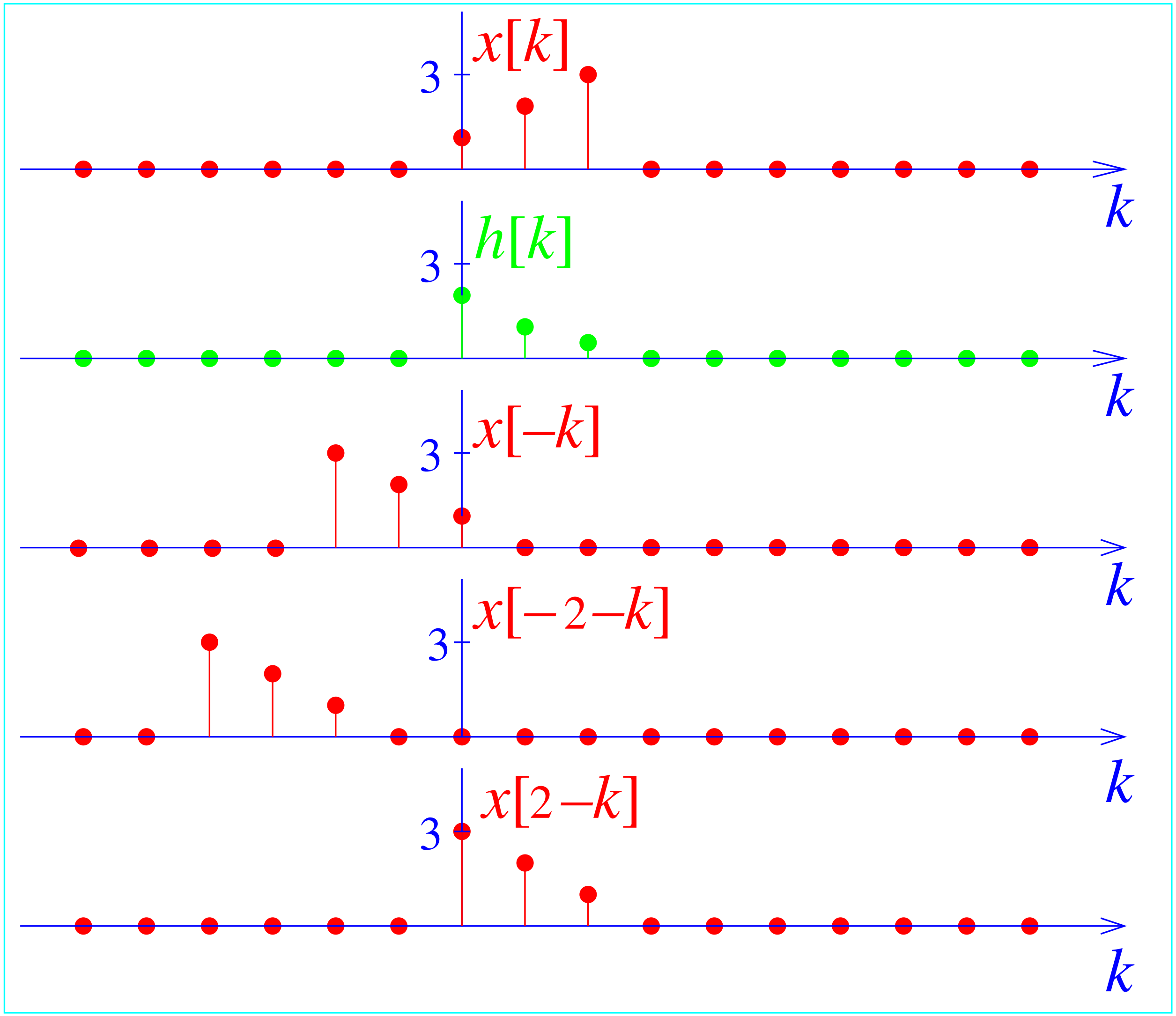
COMP ENG 4TL4:

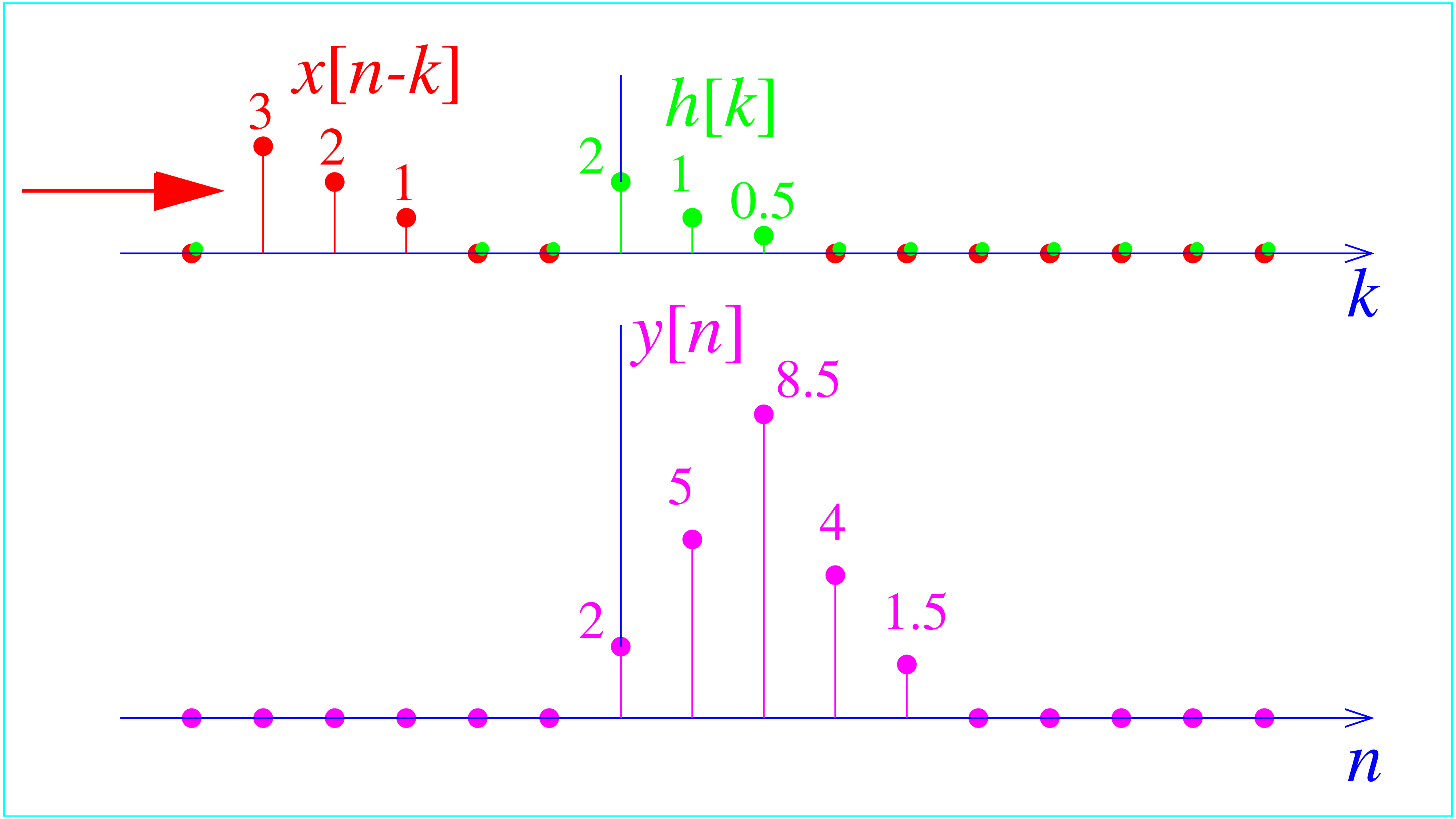
Digital Signal Processing

Notes for Lecture #6

Friday, September 19, 2003

Another example: Convolution of two sequences $x[n] = \{\ldots,0,1,2,3,0,\ldots\}$ and $h[n] = \{\ldots,0,2,1,0.5,0,\ldots\}$.




x[n-k]
3
2
1
h[k]
2
1
0.5
k
y[n]
8.5
5
4
2
1.5
n

Stability: LTI systems are stable *iff* (if and only if):

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

Proof: Let the input $x[n]$ be bounded so that $|x[n]| < L_x < \infty$, $\forall n \in [-\infty, \infty]$. Then:

$$\begin{aligned} |y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] \right| \\ &\leq \sum_{k=-\infty}^{\infty} |x[n-k]|\, |h[k]| \\ &\leq L_x \sum_{k=-\infty}^{\infty} |h[k]| \end{aligned}$$

$$\Rightarrow \quad |y[n]| < \infty \qquad \text{if} \sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

Now it remains to be proven that if:

$$\sum_{k=-\infty}^{\infty} |h[k]| = \infty,$$

then a *bounded input* can be found that will cause an *unbounded output*. Consider:

$$x[n] = \begin{cases} h^*[-n] \,/\, |h[-n]|\,, & h[n] \neq 0, \\ 0\,, & h[n] = 0, \end{cases} \quad \Rightarrow$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[-k]\,h[k] = \sum_{k=-\infty}^{\infty} |h[k]| \quad \Rightarrow$$

If $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, the output sequence is *unbounded*.

Definition: A causal system is one for which the output $y[n]$ depends on the inputs $\{\ldots, x[n-2], x[n-1], x[n]\}$ only.

Causality: An LTI system is causal *iff* its impulse response $h[n] = 0$ for $n < 0$.

Proof:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\
&= \sum_{k=0}^{\infty} h[k]\, x[n-k], \quad \text{if } h[n] = 0 \text{ for } n < 0.
\end{aligned}
$$

This equation clearly satisfies the definition given above.

Now it remains to be proven that if $h[n] \neq 0$ for $n < 0$, then the system *can be noncausal*. Let:

$$\begin{aligned} h[n] &= 0, \quad n < -1, \\ h[-1] &\neq 0, \qquad\qquad \Rightarrow \end{aligned}$$

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k] + h[-1]\, x[n+1] \qquad \Rightarrow$$

$y[n]$ depends on $x[n+1]$ $\quad\Rightarrow\quad$ the system is *noncausal*.

Example: An LTI system with:

$$h[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

1. Since $h[n] = 0$ for $n < 0$, the system is *causal*.
2. To decide on stability, we must compute the sum:

$$\begin{aligned} S &= \sum_{k=-\infty}^{\infty} |h[k]| \\ &= \sum_{k=0}^{\infty} |a|^k = \begin{cases} \frac{1}{1-|a|}, & |a| < 1, \\ \infty, & |a| \geq 1, \end{cases} \end{aligned}$$

$\Rightarrow$ the system is stable only for $|a| < 1$.

2.3 Linear Constant-Coefficient Difference (LCCD) Equations

Consider LTI systems satisfying:

$$\sum_{k=0}^{N} a[k]\, y[n-k] = \sum_{k=0}^{M} b[k]\, x[n-k]$$

ARMA

(Autoregressive Moving Average)

Particular cases:

$$y[n] = \sum_{k=0}^{M} b[k]\, x[n-k]$$

MA (Moving Average)

$$\sum_{k=0}^{N} a[k]\, y[n-k] = x[n]$$

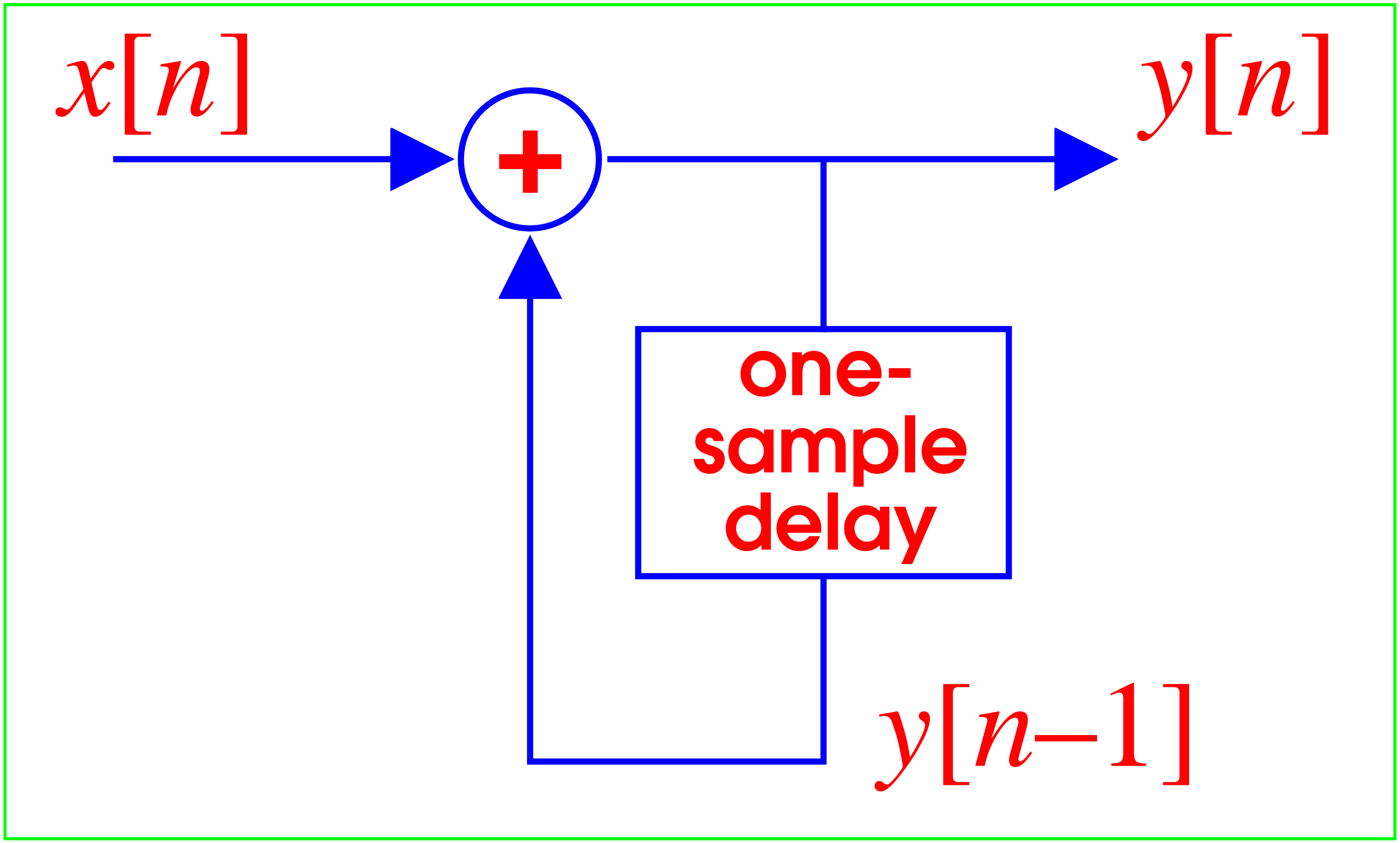
AR (Autoregressive)

Example:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad \text{accumulator}$$

$$\begin{aligned} y[n] - y[n-1] &= \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] \\ &= x[n] + \left\{ \sum_{k=-\infty}^{n-1} x[k] - \sum_{k=-\infty}^{n-1} x[k] \right\} \\ &= x[n] \end{aligned}$$

$\Rightarrow$ AR system with $a\{0\} = 1$, $a\{1\} = -1$, and $N = 1$.


x[n]
y[n]
+
one-
sample
delay
y[n–1]

<u>Property:</u> MA systems are bounded-input bounded-output (BIBO) stable, i.e.:

$$|y[n]| = \left| \sum_{k=0}^{M} b[k]\, x[n-k] \right| \leq \sum_{k=0}^{M} |b[k]|\, |x[n-k]| < \infty$$

for any bounded input $|x[n]| < \infty$ and coefficient sequence $|b[n]| < \infty$.

<u>Remark:</u> AR systems *may be* unstable. For example, the system:

$$y[n] = ay[n-1] + x[n]$$

is *unstable* for $a \geq 1$, because $y[n]$ is unbounded for bounded $x[n]$.

Property: MA systems have a finite impulse response (FIR), whereas AR systems have an infinite impulse response (IIR).

Proof for MA systems:

$$h_{\mathrm{MA}}[n] = \begin{cases} 0\,, & n < 0, \\ b[n]\,, & 0 \leq n \leq M, \\ 0\,, & n > M. \end{cases} \quad \Rightarrow \quad \text{FIR!}$$

Proof for AR systems: $y[n]$ depends on $y[n-k]$, $k = 1,\ldots,\infty$

$\Rightarrow$ $y[n]$ depends on $x[n-k]$, $k = 0,\ldots,\infty$

$\Rightarrow$ the impulse response $h_{\mathrm{AR}}[n]$ is infinite,

i.e., is in general nonzero for all $n > 0$.