COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #8

Wednesday, September 24, 2003

3. FREQUENCY-DOMAIN ANALYSIS

Terminology:

– Continuous-Time Fourier Transform (CTFT) ⇒
  continuous in time, frequency *and* amplitude

– Discrete-Time Fourier Transform (DTFT) ⇒
  discrete in time, *continuous* in frequency *and* amplitude

– Discrete Fourier Transform (DFT) ⇒
  discrete in time *and* frequency, continuous in amplitude

– Fast Fourier Transform (FFT) ⇒
  one common implementation of the DFT

Note: All of these transforms deal with *continuous* amplitudes. In practical applications of DSP, you will work with *digital* signals and processors, so you will need to consider the effects of *finite-precision arithmetic*.
See Chapter 6 of Oppenheim and Schafer for further details.

3.1 Discrete-Time Fourier Transform (DTFT)

Recall how in Lecture #2 we represented a sampled continuous-time signal $x_c(t)$ by a continuous-time modulated impulse-train signal $x_s(t)$:

$$
\begin{aligned}
x_s(t) &= \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT)\,,
\end{aligned}
$$

where $x[n] = x_c(nT)$ is the point-sampled discrete-time sequence.

Taking the Continuous-Time Fourier Transform (CTFT) of $x_s(t)$ gives:

$$\begin{aligned} X_s(j\Omega) &= \int_{-\infty}^{\infty} x_s(t)\ e^{-j\Omega t}\, dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\, \delta(t - nT)\, e^{-j\Omega t}\, dt \\ &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT)\, e^{-j\Omega t}\, dt \\ &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega nT}, \end{aligned}$$

where Ω is the continuous-time frequency in units of radians/s.

Reformulating this equation in terms of the discrete-time frequency $\omega = \Omega T$ (with units of radians) gives:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}.$$

This equation describes the Discrete-Time Fourier Transform (DTFT) of the discrete-time sequence $x[n]$.

Note: The frequency is represented by a continuous variable $\omega \Rightarrow$
the DTFT has infinite frequency resolution.

Note: $X(e^{j\omega})$ is periodic with period 2π:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{=1}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n} = X\left(e^{j(\omega+2\pi)}\right),$$

in agreement with our derivation of the Nyquist sampling theorem in Lecture #2:
$\omega = 2\pi$ radians (discrete-time)
$\to \Omega = 2\pi/T$ radians/s (continuous-time)
$\equiv f = 1/T = f_s$ Hz.

Because of the periodic nature of the DTFT, the inverse transform of the DTFT can be obtained by considering just one period of the DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\!\left(e^{j\omega}\right) e^{j\omega n}\, d\omega.$$

This equation describes the inverse DTFT of the discrete-time spectrum $X(e^{j\omega})$. It is straightforward to show that this is indeed the inverse of the DTFT:

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\pi}^{\pi} X\!\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \\
= \; & \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x[m]\; e^{j\omega(n-m)}\, d\omega \\
= \; & \sum_{m=-\infty}^{\infty} x[m]\, \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}\, d\omega}_{\delta[n-m]} \; = x[n]\,.
\end{aligned}$$

Compare continuous-time Fourier series and the DTFT:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{j\frac{2\pi n}{T}t}, \qquad X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \qquad \text{FS}$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \qquad \text{DTFT}$$

<u>Observation:</u> replacing in the Fourier series:

$$x(t) \rightarrow X\left(e^{j\omega}\right);\ \ X_n \rightarrow x[n];\ \ t \rightarrow -\omega;\ \ T \rightarrow 2\pi,$$

we obtain the DTFT!!!

(*See Jeff Bondy's slides from Tutorial #1 for a review of continuous-time Fourier series.*)

An important conclusion follows:

The DTFT is equivalent to Fourier series but applied to the “opposite” domain. In Fourier series, a periodic continuous signal is represented as a sum of exponentials weighted by discrete Fourier (spectral) coefficients. In the DTFT, a periodic continuous spectrum is represented as a sum of exponentials weighted by discrete signal values.

Remarks:

– The DTFT can be derived directly from Fourier series.

– All developments for Fourier series can be applied to the DTFT.

– The relationship between Fourier series and the DTFT illustrates the duality between the time and frequency domains.

Properties of “Fourier spectra”:

– In general, Fourier spectra are complex-valued: X_n for Fourier series; $X(j\Omega)$ for the CTFT; $X(e^{j\omega})$ for the DTFT

  Consequently, we sometimes refer to the *magnitude* (spectrum), e.g., $|X(e^{j\omega})|$, or the *phase* (spectrum), e.g., $\angle X(e^{j\omega})$.

– For real signals: $X_{-n} = X_n{}^*$ for Fourier series; $X(-j\Omega) = X^*(j\Omega)$ for the CTFT; $X(e^{-j\omega}) = X^*(e^{j\omega})$ for the DTFT

  That is, the magnitude spectrum is mirrored in the y-axis, e.g., $|X(e^{-j\omega})| = |X(e^{j\omega})|$, while the phase spectrum is mirrored in the y-axis *and* the x-axis, e.g., $\angle X(e^{-j\omega}) = -\angle X(e^{j\omega})$.

Parseval theorem for the DTFT:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Proof:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n]\, x^*[m]\, e^{-j\omega(n-m)}\, d\omega$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n]\, x^*[m]\, \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(n-m)}\, d\omega}_{\delta[n-m]}$$

$$= \sum_{n=-\infty}^{\infty} |x[n]|^2 .$$

When does the DTFT exist ($|X(e^{j\omega})| < \infty$)?

Sufficient condition:

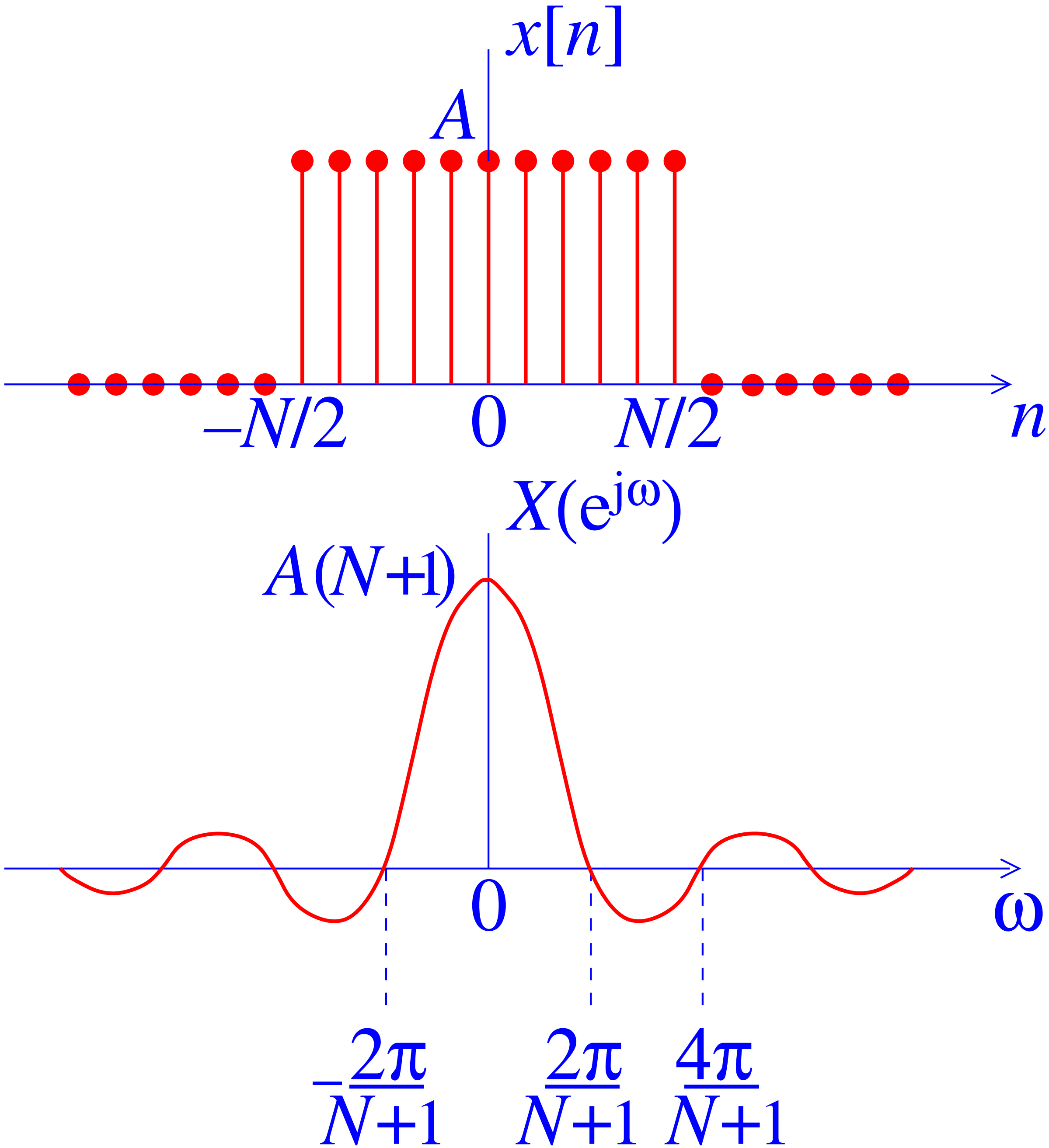
$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Proof:

$$\begin{aligned}
\left|X\left(e^{j\omega}\right)\right| &= \left|\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\right| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]|\, \underbrace{\left|e^{-j\omega n}\right|}_{=1} \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty.
\end{aligned}$$

Example: Finite-energy rectangular signal:

$$
\begin{aligned}
X\left(e^{j\omega}\right) &= \sum_{n=-N/2}^{N/2} A e^{-j\omega n} = A \sum_{n=-N/2}^{N/2} e^{-j\omega n} \\
&= A\,(N+1)\,\frac{\sin\left(\frac{N+1}{2}\,\omega\right)}{(N+1)\sin\left(\frac{\omega}{2}\right)} \\
&\simeq A\,(N+1)\,\underbrace{\frac{\sin\left(\frac{N+1}{2}\,\omega\right)}{\frac{N+1}{2}\,\omega}}_{\text{“sinc function”}} \qquad \text{for } \omega \ll \pi
\end{aligned}
$$


x[n]
A
–N/2
0
N/2
n
X(e^jω)
A(N+1)
0
ω
–2π/(N+1)
2π/(N+1)
4π/(N+1)