Computer Engineering 4TL4: Digital Signal Processing (Fall 2003)

Solutions to Midterm Exam

1. **Consider a discrete-time linear time-invariant system with impulse response $h[n]$. Show that if the input $x[n]$ is a periodic sequence with period *N* (i.e., if $x[n]=x[n+N]$), then the output $y[n]$ is also a periodic sequence with period *N*. (20 pts)**

One approach to showing this property is via discrete-time convolution. The output $y[n]$ is the linear convolution of $h[n]$ and $x[n]$:

$$
\begin{aligned}
y[n] &= x[n]*h[n] \\
&= \sum_{k=-\infty}^{\infty} x[n-k]h[k].
\end{aligned}
$$

Let $n = m + N$:

$$
\begin{aligned}
y[m+N] &= \sum_{k=-\infty}^{\infty} x[(m+N)-k]h[k] \\
&= \sum_{k=-\infty}^{\infty} x[(m-k)+N]h[k].
\end{aligned}
$$

Since $x[n]$ is periodic, $x[n]=x[n+rN]$ for any integer r. Hence:

$$
\begin{aligned}
y[m+N] &= \sum_{k=-\infty}^{\infty} x[m-k]h[k] \\
&= y[m].
\end{aligned}
$$

A second time-domain approach is to break $x[n]$ into an infinite sum of the finite sequence $\tilde{x}[n]$ of length N shifted by integer multiples of N samples:

$$
\tilde{x}[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\Rightarrow x[n] = \sum_{k=-\infty}^{\infty} \tilde{x}[n+kN].
$$

The linearity of the system allows us to compute the output of this system as the sum of the outputs to each of the finite inputs:

$$
\begin{aligned}
y[n] &= \mathcal{T}\left\{\sum_{k=-\infty}^{\infty} \tilde{x}[n+kN]\right\} \\
&= \sum_{k=-\infty}^{\infty} \mathcal{T}\{\tilde{x}[n+kN]\},
\end{aligned}
$$

where $\mathcal{T}\{\cdot\}$ indicates the linear transformation of the system.

The time-invariance of the system means that the output for the each of these time-shifted finite sequences is just a time-shifted version of the output for the finite sequence $\tilde{x}[n]$. Letting $\tilde{y}[n] = \mathcal{T}\{\tilde{x}[n]\}$:

$$y[n] = \sum_{k=-\infty}^{\infty} \tilde{y}[n+kN],$$

that is, a periodic sequence with period *N*.

A third approach is to use the Fourier-domain or *z*-domain transfer functions, $H(e^{j\omega})$ and $H(z)$ respectively. The discrete-time convolution in the time domain becomes multiplication in the Fourier and *z*-domains:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad \text{and} \qquad Y(z) = H(z)X(z).$$

Multiplying both side of each equation by $e^{j\omega N}$ and z^N, respectively, gives:

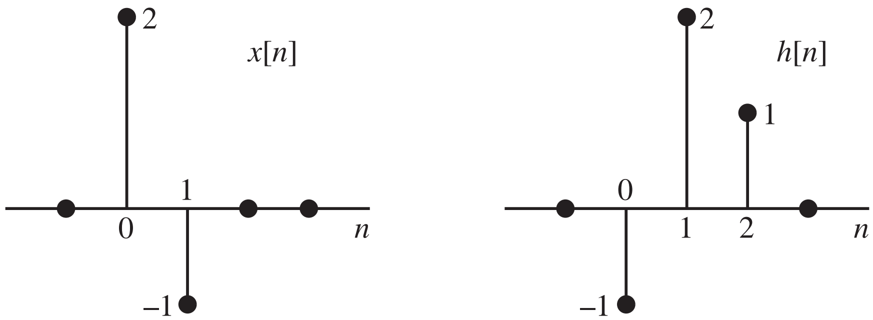
$$e^{j\omega N}Y(e^{j\omega}) = H(e^{j\omega})e^{j\omega N}X(e^{j\omega}) \qquad \text{and} \qquad z^N Y(z) = H(z)z^N X(z).$$

Using the time-shift properties of the Fourier and *z*-transforms, the terms $e^{j\omega N}X(e^{j\omega})$ and $z^N X(z)$ are the Fourier and *z*-transforms, respectively, of $x[n+N]$. Since $x[n]$ is periodic:

$$\begin{aligned} e^{j\omega N}Y(e^{j\omega}) &= H(e^{j\omega})e^{j\omega N}X(e^{j\omega}) \\ &= H(e^{j\omega})X(e^{j\omega}) \\ &= Y(e^{j\omega}), \end{aligned} \qquad \text{and} \qquad \begin{aligned} z^N Y(z) &= H(z)z^N X(z) \\ &= H(z)X(z) \\ &= Y(z). \end{aligned}$$

Taking either of the inverse transforms gives $y[n+N] = y[n]$, i.e., $y[n]$ is periodic with period *N*.

2. Consider a system having the impulse response $h[n]$ and the input sequence $x[n]$ shown in the figure below.

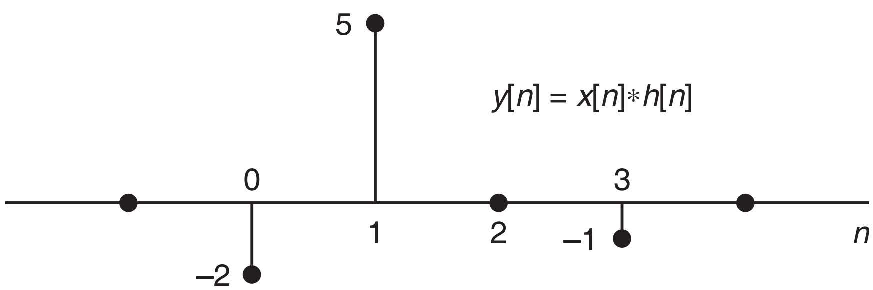


a. Use discrete-time *linear* convolution to find the system's response to this input.

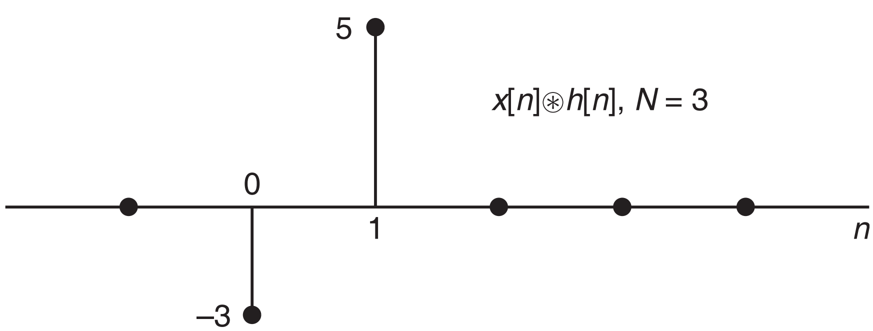
b. Find the 3-point *circular* convolution of $x[n]$ (for $n = 0, 1, 2$) with $h[n]$ (for $n = 0, 1, 2$).

c. To what length N do you need to zero-pad $h[n]$ and $x[n]$ so that time-aliasing is avoided, i.e., the N-point circular convolution is identical to the linear convolution from part a? (20 pts)

a. The system's response to this input is shown in the figure below.



b. The 3-point circular convolution of $x[n]$ with $h[n]$ is shown in the figure below.



c. The N-point circular convolution is identical to the linear convolution from part a, for $N \geq L + P - 1$, where L and P are the lengths of the two convolved sequences. In this case where L = 2 and P = 3, to avoid time-aliasing we must take at least a 4-point circular convolution.

3. **In Lab #1 you wrote a MATLAB function to implement a 3-bit uniform rounding quantizer, which did not produce a very high-fidelity output signal. Three options available to improve the quantizer *for the particular speech signal used in Lab #1* are:**

   i. **Increase the number of quantization levels;**

   ii. **Change to an optimal nonuniform quantizer; or**

   iii. **Pass the signal through a μ-law compander before performing the uniform quantization.**

   Explain the advantages and disadvantages of each of these choices, and compare and contrast them with each other (i.e., describe the advantages and disadvantages relative to the other options).

   (20 pts)

   Option i has the advantage over the other two options in that we can still utilize our very simple uniform quantizer. The disadvantage is the increased number of bits required to code the greater number of quantization levels. This will necessitate greater computational resources, and if the quantizer is in an A/D converter, then the conversion process will either require more comparators (e.g., flash A/D) or more time (e.g., successive approximation A/D).

   In option ii we can make use of the known *amplitude distribution* of the particular speech signal to quantize infrequently-occurring amplitudes with lower precision (i.e., greater quantizer error) than frequently-occurring amplitudes. The frequently-occurring amplitudes contribute more to the perception of the speech signal, so we are able to code the signal with fewer bits than is possible with options i and iii (for a given signal fidelity). The disadvantage is that we need to calculate the amplitude distribution and then design and implement an optimal nonuniform quantizer for that particular distribution.

   For option iii we can utilize our knowledge that speech signals have very peaked amplitude distributions (typically referred to as Laplacian), and consequently a compressive function such as the μ-law compander will produce a more uniform amplitude distribution going into the uniform quantizer. The quantization will still be suboptimal, i.e., a greater number of quantization levels will be required to obtain the same fidelity as option ii, but it will be closer to optimal than the original quantizer. The μ-law compander adds only a little complexity to the design of the quantizer—much less than the added complexity of option iii, because we are not required to calculate the amplitude distribution or to design and implement a nonuniform quantization scheme.

   Note that in this question we have a known signal, so we can set the full-scale range for all of these quantizers such that they avoid saturation. That is, peak clipping is not an issue in comparing these three options.

4. A linear time-invariant system is described by the difference equation:

$$y[n] = x[n] + 2x[n-1] + x[n-2].$$

a. Determine $h[n]$, the impulse response of the system.

b. Is this a stable system? Justify your answer.

c. Determine $H(e^{j\omega})$, the frequency response of the system. Use trigonometric identities to obtain a simple expression for $H(e^{j\omega})$.

d. Sketch the magnitude and phase responses of this system. (20 pts)

a. The impulse response of this system is:

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

b. This system is *stable* because the impulse response is absolutely summable. This is always the case for an FIR system that has $|h[n]| < \infty, \forall n$.

c. Taking the Fourier transform of each of the terms in $h[n]$ gives:

$$\begin{aligned} H(e^{j\omega}) &= 1 + 2e^{-j\omega} + e^{-2j\omega} \\ &= e^{-j\omega}\{e^{j\omega} + 2 + e^{-j\omega}\} \\ &= e^{-j\omega}\{2 + (e^{j\omega} + e^{-j\omega})\} \\ &= e^{-j\omega}\{2 + 2\cos(\omega)\} \\ &= e^{-j\omega}2\{1 + \cos(\omega)\}. \end{aligned}$$

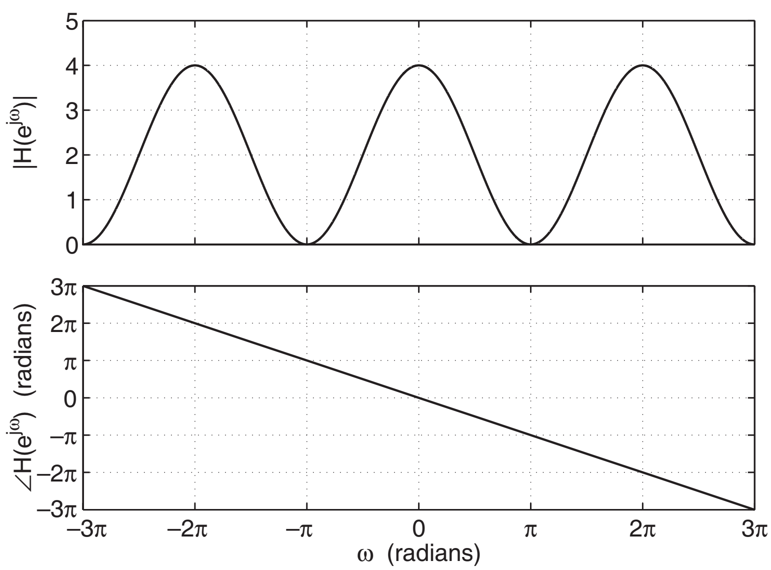
d. The magnitude response:

$$\begin{aligned} |H(e^{j\omega})| &= |e^{-j\omega}2\{1+\cos(\omega)\}| \\ &= 2\{1+\cos(\omega)\}, \end{aligned}$$

and phase response:

$$\begin{aligned} \angle H(e^{j\omega}) &= \angle e^{-j\omega}2\{1+\cos(\omega)\} \\ &= -\omega, \end{aligned}$$

are shown in the figure to the right.



5. The continuous-time signal:

$$x_c(t) = \cos(4000\pi t), \quad -\infty < t < \infty,$$

is sampled with sampling period *T* to obtain the discrete-time signal:

$$x[n] = \cos\left(\frac{\pi}{3}n\right), \qquad -\infty < n < \infty.$$

a. Determine a choice for *T* that is consistent with this information.

b. Is your choice for *T* unique? If so, explain why. If not, specify another choice of *T* that is consistent with the information given. (20 pts)

a. Since $x[n] = x_c(nT)$:

$$\frac{\pi}{3}n = 4000\pi nT$$

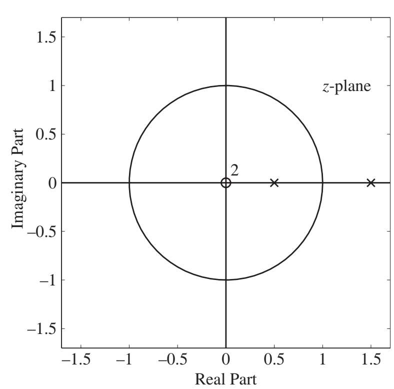
$$\Rightarrow \quad T = \frac{\pi}{3\cdot 4000\pi} = \frac{1}{12000} \text{ seconds.}$$

b. This choice of *T* is not unique. For example, since:

$$\cos\left(\frac{\pi}{3}n\right) = \cos\left(\frac{\pi}{3}n + 2\pi n\right) = \cos\left(\frac{7\pi}{3}n\right),$$

another choice that satisfies the equality $x[n] = x_c(nT)$ is $T = 7/12000$ seconds.

6. The *z*-domain transfer function $H(z)$ of a system has the pole-zero plot shown below.

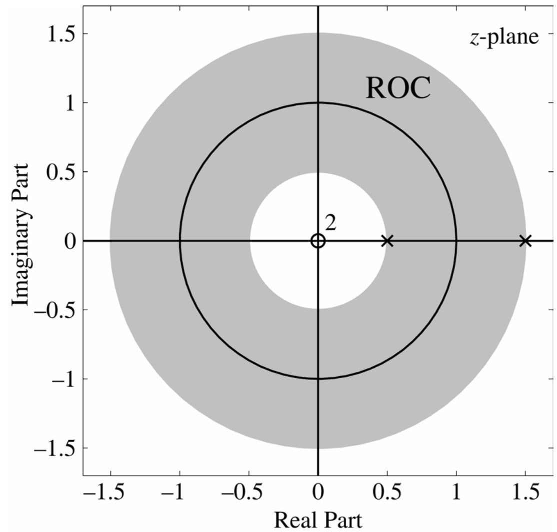


a. Sketch the transfer function's region of convergence (ROC) if this system is known to be *stable*. What can you say about the system's impulse response $h[n]$ in this case?

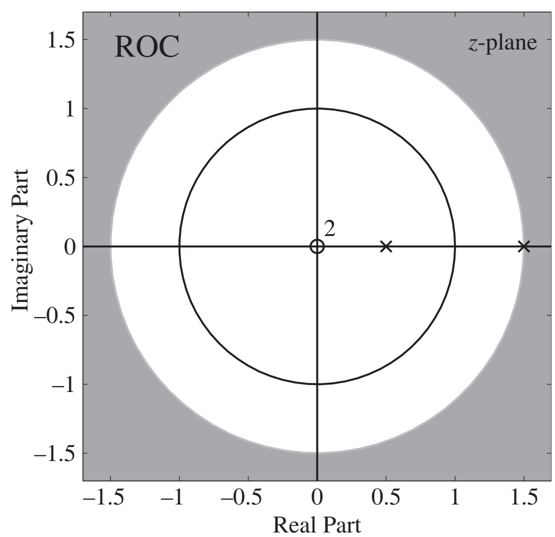
b. Sketch the transfer function's ROC if this system is known to be *causal*. What can you say about the system's impulse response $h[n]$ in this case?

c. Sketch the transfer function's ROC if this system is known to be both *stable and causal*. What can you say about the system's impulse response $h[n]$ in this case? (20 pts)

a. For the system to be stable, the ROC must include the unit circle. The only possible ROC, shown in the figure below, is for a *two-sided* impulse response $h[n]$ in this case. Consequently, the system is noncausal.



b. For the system to be causal, the ROC must include $z = \infty$. The only possible ROC, shown in the figure below, is for a *right-sided* impulse response $h[n]$ in this case. In this case the ROC does not include the unit circle, so the system is unstable.



c. No ROC exists that can satisfy the requirements of both stability and causality for this transfer function–either the system is *stable and noncausal* or it is *unstable and causal*. Therefore, no impulse response exists for this transfer function that satisfies both of these requirements.