Fourier Series/Transform Review

Fourier Review

Fourier Series and Transforms try to form a signal out of sinusoids. These sinusoids have a specific frequency and go on forever. That is your nice time series which is represented by points in time will now be represented by points in frequency. This is why we use the terms "Fourier domain" and "frequency domain" interchangeably.

Reminder:

$$ae^{(jbt)} = a\cos(bt) + ja\sin(bt)$$

What Transform, When?

Start Domain	Discrete or Continuous	Periodic	Transform
Time	Discrete	Yes	DTFS
Time	Discrete	No	DTFT
Time	Continuous	Yes	FS
Time	Continuous	No	FT
Frequency	Discrete	Yes	I-DTFS
Frequency	Discrete	No	I-FS
Frequency	Continuous	Yes	I-DTFT
Frequency	Continuous	No	I-FT

Discrete Time Fourier Series

DTFS: $$X[k] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\Omega_0 n}$$

I-DTFS: $$x[n] = \frac{1}{N}\sum_{k=<N>} X[k]e^{jk\Omega_0 n}$$

X[k] and x[n] have period N

$\Omega_0 = 2\pi/N$

Discrete Time Fourier Transform

DTFT: $$X[e^{j\Omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

I-DTFS: $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$$

X[k] has period 2π

Fourier Series

FS: $$X[k] = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}\,dt$$

I-FS: $$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$$

X(t) has period T

$\Omega_0 = 2\pi/T$

Fourier Transform

FT: $$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

I-FT: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

The Fourier Transform is the general transform, it can handle periodic and non-periodic signals. For a periodic signal it can be thought of as a transformation of the Fourier Series

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - n\omega_0)$$

Fourier Series

$$X[k] = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$$

$$X[k] = \underbrace{\frac{1}{T}\int_{<T>} x(t)\cos(\omega_o kt)dt}_{A_k} - j\underbrace{\frac{1}{T}\int_{<T>} x(t)\sin(\omega_o kt)dt}_{B_k}$$

$$X[k] = \underbrace{\sqrt{A_k^2 + B_k^2}}_{|X_k|}\,\underbrace{e^{-\tan^{-1}\left(B_k/A_k\right)}}_{e^{\theta_k}} = |X_k|e^{\theta_k}$$

Fourier Series – Real Signals

$$X[k] = \underbrace{\frac{1}{T}\int_{<T>} x(t)\cos(\omega_o kt)dt}_{A_k} - j\underbrace{\frac{1}{T}\int_{<T>} x(t)\sin(\omega_o kt)dt}_{B_k}$$

If x(t) is real valued: $A_k = A_{-k}$ $B_k = -B_{-k}$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t} = X[0] + \sum_{k=1}^{\infty}\left(X[k]e^{jk\omega_o t} + X[-k]e^{-jk\omega_o t}\right) = X[0] + \sum_{k=1}^{\infty}\left((A_k + jB_k)e^{jk\omega_o t} + (A_{-k} + jB_{-k})e^{-jk\omega_o t}\right)$$

$$x(t) = X[0] + \sum_{k=1}^{\infty}\left((A_k + jB_k)e^{jk\omega_o t} + (A_k - jB_k)e^{-jk\omega_o t}\right) = X[0] + \sum_{k=1}^{\infty}\left(A_k\left(e^{jk\omega_o t} + e^{-jk\omega_o t}\right) + jB_k\left(e^{jk\omega_o t} + e^{-jk\omega_o t}\right)\right)$$

$$x(t) = X[0] + 2\sum_{k=1}^{\infty}\left(A_k\cos(k\omega_o t) + B_k\sin(k\omega_o t)\right) = X[0] + 2\sum_{k=1}^{\infty}\mathrm{Re}\left(X[k]e^{jk\omega_o t}\right)$$

$$x(t) = X[0] + 2\sum_{k=1}^{\infty}\mathrm{Re}\left(|X[k]|e^{j\theta_k}e^{jk\omega_o t}\right) = X[0] + 2\sum_{k=1}^{\infty}|X[k]|\cos(k\omega_0 t + \theta_k)$$

Fourier Series – Real +Even/Odd

$$x(t) = X[0] + 2\sum_{k=1}^{\infty}\mathrm{Re}\left(|X[k]|e^{j\theta_k}e^{jk\omega_o t}\right)$$

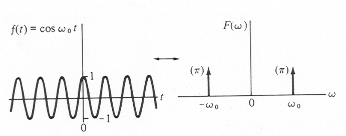
$$x(t) = X[0] + 2\sum_{k=1}^{\infty}\mathrm{Re}\left\{(A_k - jB_k)(\cos(k\omega_o t) + j\sin(k\omega_o t))\right\}$$

$$x(t) = X[0] + 2\sum_{k=1}^{\infty}\left(A_k\cos(k\omega_o t) + B_k\sin(k\omega_o t)\right)$$

Even: f(t) = f(-t), therefore $B_k = 0$; Cosine Series

Odd: f(t) = -f(-t), therefore $A_k = 0$; Sine Series

Cosine Fourier Series



$$f(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

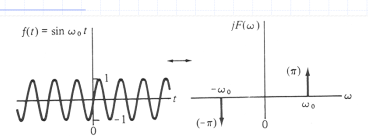
Even Function

FS $X[1] = X[-1] = 1/2$

FT = 2π(FS) $X(j\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$

When is FT the continuous counterpart to 2πFS?
How do the Delta's move as frequency changes?

Sine Fourier Transform



$$f(t) = \sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

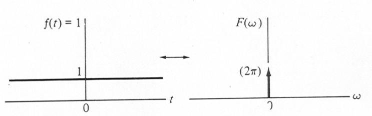
Odd Function

FS $X[1] = -X[-1] = 1/2j$

FT = 2π(FS) $X(j\omega) = j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0)$

The Fourier Transform of an Odd Signal is Odd.
Notice the Fourier Domain graph is in jF(ω). It is imaginary.

DC Fourier Transform



$$f(t) = 1 = e^{j\omega_0 t}\ ; \qquad \omega_0 = 0$$

DC Function

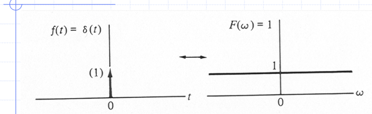
FS $X[0] = 1$

FT ↔ (FS) $X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$

FT $X(j\omega) = 2\pi\delta(\omega)$

The FT of a signal with a DC component is separable.
The DC component of a time signal is statistically the MEAN.

Delta Fourier Transform



$$f(t) = \delta(0)$$

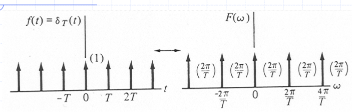
Delta Function

FS - No Fourier Series, Not Periodic

FT $X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi kt}dt = e^{-j2\pi k(0)} = 1$

The FT is only congruent with the FS for PERIODIC signals.
A delta has an infinitely steep rise time, therefore it has a great deal of high frequencies

Pulse Train Fourier Transform



$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

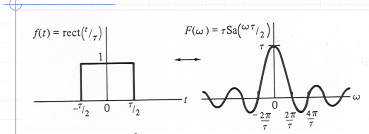
Function with Period T

FS $X[k] = 1/T$ $\quad$ *for all k*

$$X(j\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - nT)e^{-j2\pi kt}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j2\pi knT}dt = 2\pi/T \sum_{k=-\infty}^{\infty} \delta\left(\omega - 2\pi k/T\right)$$

What happens in the Frequency Domain when the time between pulses is shortened? When T → 0? When T = 0?

Time Window Fourier Transform



$$f(t) = \begin{cases} 1, & |t| < \tau/2 \\ 0, & |t| \geq \tau/2 \end{cases} \equiv rect\left(t/\tau\right)$$

Not Periodic – No FS

FT $X(j\omega) = \dfrac{2\sin\omega\tau/2}{\omega} = \tau\, Sinc\left(\omega\tau/2\right)$

$$Sinc(x) \equiv Sa(x) \equiv \frac{\sin(x)}{x}$$

Ideal Filter Fourier Transform

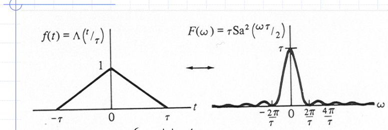
$$x(t) = Sinc(Wt)$$

Not Periodic – No FS

FT $X(j\omega) = \begin{cases} \pi/W, & |\omega| < W \\ 0, & |\omega| \geq W \end{cases} = \dfrac{\pi}{W} rect\left(\omega/2W\right)$

Why is this called the "ideal filter"?
Notice similarities between this and rectangular time window, and how W here is a counterpart to τ there in controlling width.

Triangle Fourier Transform



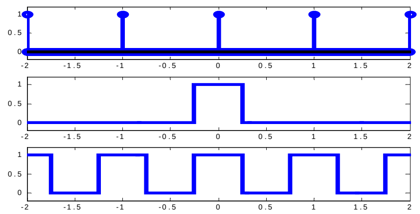
$$x(t) = \begin{cases} 1 - |t|/\tau, & |t| < \tau \\ 0, & |t| \geq \tau \end{cases} \equiv \Lambda\left(t/\tau\right)$$

Not Periodic – No FS

FT $X(j\omega) = \tau\left[Sinc\left(\omega\tau/2\right)\right]^2$

Sinc squared can never be negative. Why are we introducing these signals? They are the foundation of most analog communication signals.

More Complex Example



An pulse train with period (T) one second is convolved with a time windowing function with timing (τ) of 0.5 seconds, to produce a 50% duty cycle square wave.

More Complex Example

The spectrum of the pulse train is:

$$X_1(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

The spectrum of the square-wave is:

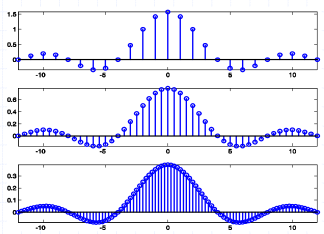
$$X_2(j\omega) = \tau \; Sinc\left(\frac{\omega\tau}{2}\right)$$

Convolution turns into Multiplication in the Freq Domain

$$X_1(j\omega) \cdot X_2(j\omega) = \frac{2\pi\tau}{T} Sinc\left(\frac{\omega\tau}{2}\right) \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

This turns into a line spectra, and how it changes with changing the parameters is very informative

Constant τ $\tau = 0.5$



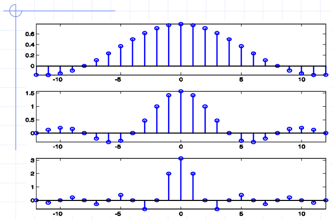
T = 2

T = 4

T = 8

- Amplitude DECREASES as 1/T
- Line spectra resolution INCREASES as T
- The envelope is INDEPENDENT of T

Constant T T = 2



$\tau = 0.25$

$\tau = 0.5$

$\tau = 1$

- Amplitude INCREASES in proportion to Tau
- Line spectra resolution is INDEPENDENT of Tau
- The spectrum SPREADS as the window shortens

!!! TIME RESOLUTION AND FREQUENCY RESOLUTION ARE INVERSELY RELATED !!!!!!!!