EE 4CL4 – Control System Design

Solutions to Homework Assignment #2

1. **Consider an electronic amplifier with input voltage $v_i(t)$ and output voltage $v_o(t)$. Assume that:**

$$v_o(t) = 8v_i(t) + 2\,. \tag{1}$$

   a. **Show that the amplifier does not strictly satisfy the principle of superposition. Thus, this system is not strictly linear. (A better term for this system would be affine.)**

   b. **Note that the system can also be written as follows:**

$$v_o(t) = 8v_i(t) + 2d_i(t), \tag{2}$$

   where $d_i(t)$ is a constant offset (equal to 1).
   Show that the principle of superposition does hold for the input vector $[v_i(t)\ \ d_i(t)]^T$.

   c. **Obtain an incremental model for $\Delta v_o(t) = v_o(t) - v_{oQ}$, $\Delta v_i(t) = v_i(t) - v_{iQ}$, where (v_{iQ}, v_{oQ}) is any point satisfying the model given by Eq. (1) above. Show that this incremental model is the same for all choices of the pair (v_{iQ}, v_{oQ}).** **(25 pts)**

   a. First consider two different (independent) inputs $v_{i1}(t)$ and $v_{i2}(t)$. The corresponding outputs would then be $v_{o1}(t) = 8v_{i1}(t)+2$ and $v_{o2}(t) = 8v_{i2}(t)+2$. To test superposition we consider now one input given by $v_i(t) = v_{i1}(t)+ v_{i2}(t)$. Equation (1) implies that the output is given by:

$$v_o(t) = 8(v_{i1}(t) + v_{i2}(t)) + 2 = 8\,v_{i1}(t) + 8\,v_{i2}(t) + 2. \tag{1$'$}$$

   However, superposition implies that v_o(t) should be equal to $v_{o1}(t)+v_{o2}(t)$, i.e. equal to $8\,v_{i1}(t) + 8\,v_{i2}(t) + 4$. This is different to the result given in Eq. (1′). Hence the property of superposition does not hold for this system.

   b. The system model can be written as $v_o(t) = \alpha\,(u(t))^T : \alpha = [8\ 2]$; $u(t) = [v_i(t)\ d_i(t)]$. If we consider now two vector inputs $u_1(t)$ and u_2(t), it is straightforward to check that the system response to the input $u_1(t)+u_2(t)$ according to Eq. (2) can also be obtained by adding up the output to $u_1(t)$ and the output to $u_2(t)$. Thus superposition holds.

   c. If (v_{iQ}, v_{oQ}) describes an operating point, then (v_{iQ}, v_{oQ}) satisfies Eq. (1), i.e. $v_{oQ} = 8v_{iQ} + 2$. Subtracting this from Eq. (1) we have that $v_o(t) - v_{oQ} = \Delta v_o(t) = 8(v_i(t) - v_{iQ}) = 8\Delta v_i(t)$. Thus, the incremental model is linear and it does not depend on the operating point.

2. **Consider the following nonlinear state space model:**

$$\dot{x}_1(t) = -2x_1(t) + 0.1x_1(t)x_2(t) + u(t), \tag{3}$$

$$\dot{x}_2(t) = -x_1(t) - 2x_2(t)(x_1(t))^2, \tag{4}$$

$$y(t) = x_1(t) + (1 + x_2(t))^2\,. \tag{5}$$

   Build a linearized model around the operating point given by $u_Q = 1$. **(25 pts)**

To linearize a state space model with one input variable $u(t)$, two state variables $x_1(t)$ and $x_2(t)$ and one output variable $y(t)$, i.e.,:

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}f_1(\mathbf{x}(t),u(t))\\ f_2(\mathbf{x}(t),u(t))\end{bmatrix},$$

$$y(t)=g(\mathbf{x}(t),u(t)),$$

around an operating point (u_Q, x_{1Q}, x_{2Q}, y_Q), we define four new variables $\Delta u(t) = u(t) - u_Q$, $\Delta x_1(t) = x_1(t) - x_{1Q}$, $\Delta x_2(t) = x_2(t) - x_{2Q}$, and $\Delta y(t) = y(t) - y_Q$, and use the following approximations:

$$\Delta\dot{\mathbf{x}}(t)=\begin{bmatrix}\Delta\dot{x}_1(t)\\ \Delta\dot{x}_2(t)\end{bmatrix}\approx\begin{bmatrix}\left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}}\\ \left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}}\end{bmatrix}\begin{bmatrix}\Delta x_1(t)\\ \Delta x_2(t)\end{bmatrix}+\begin{bmatrix}\left.\frac{\partial f_1}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}}\\ \left.\frac{\partial f_2}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}}\end{bmatrix}\Delta u(t),$$

$$\Delta y(t)\approx\begin{bmatrix}\left.\frac{\partial g}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}} & \left.\frac{\partial g}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}}\end{bmatrix}\begin{bmatrix}\Delta x_1(t)\\ \Delta x_2(t)\end{bmatrix}+\left.\frac{\partial g}{\partial u}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\u=u_Q}}\Delta u(t).$$

The linearized model for a (general) operating point is given by:

$$\mathrm{d}\Delta x_1(t)/\mathrm{d}t = -2\Delta x_1(t) + 0.1x_{1Q}\Delta x_2(t) + 0.1x_{2Q}\Delta x_1(t) + \Delta u(t) \tag{3′}$$

$$\mathrm{d}\Delta x_2(t)/\mathrm{d}t = (-1 - 4x_{1Q}x_{2Q})\Delta x_1(t) - 2\,(x_{1Q})^2\,\Delta x_2(t) \tag{4′}$$

$$\Delta y(t) = \Delta x_1(t) + 2(1 + x_{2Q})\Delta x_2(t) \tag{5′}$$

To obtain the linearized model for $u_Q = 1$, we compute the operating point from the system state space model (Eqs. (3)–(5)):

$$0 = -2x_{1Q} + 0.1x_{1Q}x_{2Q} + u_Q \tag{3″}$$

$$0 = -\,x_{1Q} + -2x_{2Q}\,(x_{1Q})^2 \tag{4″}$$

$$y_Q = x_{1Q} + (1 + x_{2Q})^2 \tag{5″}$$

We see that although $x_{1Q} = 0$ satisfies Eq. (4″), it does not satisfy Eq. (3″) (for $u_Q = 1$). Thus the alternative solution from (4″) requires that $x_{1Q}x_{2Q} = -0.5$. Using this value in (3″) we obtain $x_{1Q} = 0.475$ and $x_{2Q} = -1.053$. Entering these values into Eqs. (3′)–(5′) gives:

$$\mathrm{d}\Delta x_1(t)/\mathrm{d}t = -2.1053\Delta x_1(t) + 0.0475\Delta x_2(t) + \Delta u(t) \tag{3‴}$$

$$\mathrm{d}\Delta x_2(t)/\mathrm{d}t = \Delta x_1(t) - 0.4512\Delta x_2(t) \tag{4‴}$$

$$\Delta y(t) = \Delta x_1(t) - 0.1053\Delta x_2(t) \tag{5‴}$$

3. **A system transfer function is given by:**

$$H(s)=\frac{-s+1}{(s+1)^2}. \tag{6}$$

Compute the time instant, t_u, at which the step response exhibits maximum undershoot. (25 pts)

We observe that, when maximum undershoot occurs, the time derivative of the step response must vanish. Thus, we consider, $h(t)$, the impulse response instead (which is the derivative of the step response).

$$h(t)=\mathcal{L}^{-1}[H(s)]=\mathcal{L}^{-1}\left[\frac{-s+1}{(s+1)^2}\right]=\mathcal{L}^{-1}\left[\frac{-s-1}{(s+1)^2}+\frac{2}{(s+1)^2}\right]=-\mathrm{e}^{-t}+2t\,\mathrm{e}^{-t}.$$

We observe that only two positive real values for t make this signal vanish, t = 0.5 and $t = \infty$. Obviously the maximum undershoot occurs at t_u = 0.5 [s].

Alternatively, we can recall that if $H(s)$ is the system transfer function, then the Laplace transformation of its unit step response $y(t)$ is $Y(s) = H(s)/s$. $y(t)$ can be obtained by taking the inverse Laplace transform of $Y(s)$. $h(t)$ can then be computed as the time derivative of $y(t)\mu(t)$ and the value for t_u found as above.

4. **The unit step response of a system with zero initial conditions is given by:**

$$y(t)=3-2\,\mathrm{e}^{-2t}-\mathrm{e}^{-3t} \quad \forall t\geq 0. \tag{7}$$

a. **Compute the system transfer function.**

b. **Compute the system response to a unit impulse. (25 pts)**

We first recall that if $H(s)$ is the system transfer function, then the Laplace transformation of its unit step response is $Y(s) = H(s)/s$.

a. From the expression for $y(t)$ in Eq. (7) we have that $Y(s)=\frac{3}{s}-\frac{2}{s+2}-\frac{1}{s+3}=\frac{7s+18}{s(s+2)(s+3)}$.

Then $H(s)=sY(s)=\frac{7s+18}{(s+2)(s+3)}$.

b. The system unit impulse response (with zero initial conditions), $h(t)$, is equal to the inverse Laplace transform of the system transfer function, i.e.,

$$h(t)=\mathcal{L}^{-1}[H(s)]=\mathcal{L}^{-1}\left[\frac{7s+18}{(s+2)(s+3)}\right]=\mathcal{L}^{-1}\left[\frac{4}{(s+2)}+\frac{3}{(s+3)}\right]=4\,\mathrm{e}^{-2t}+3\,\mathrm{e}^{-3t}.$$

Alternatively, $h(t)$ can be computed as the time derivative of $y(t)\mu(t)$. The answer to part a. can then be obtained by taking the Laplace transform of $h(t)$.