EE 4CL4 – Control System Design

Solutions to Homework Assignment #3

1. **A discrete-time system with input $u[k]$ and output $y[k]$ is described by the difference equation:**

$$y[k]-0.8y[k-1]+0.15y[k-2]=0.2u[k-i].$$

   a. **Build a state space model for $i = 0$.**

   b. **Repeat for $i = 1$.** **(25 pts)**

   a. For $i = 0$ we can choose $x_1[k] = y[k-2]$ and $x_2[k] = y[k-1]$, thus the state space model is:

$$x_1[k+1] = x_2[k]$$

$$x_2[k+1] = -0.15x_1[k] + 0.8x_2[\mathrm{k}] + 0.2u[k]$$

$$y[k] = -0.15x_1[k] + 0.8\,x_2[k] + 0.2u[k]$$

   b. When $i = 1$ we have to add a new state variable, since the system output depends on a delayed input. We thus choose $x_3[k] = u[k-1]$ and the state space model becomes:

$$x_1[k+1] = x_2[k]$$

$$x_2[k+1] = -0.15x_1[k] + 0.8x_2[\mathrm{k}] + 0.2x_3[k]$$

$$x_3[k+1] = u[k]$$

$$y[k] = -0.15x_1[k] + 0.8\,x_2[k] + 0.2x_3[k]$$

2. **Consider a feedback control loop of a plant with nominal model $G_o(s)=\dfrac{1}{(s+1)}$. Assume that the controller $C(s)$ is such that the complementary sensitivity is:**

$$T_o(s)=\frac{4}{(s+2)^2}.$$

   a. **Show that the control loop is internally stable.**

   b. **Compute the controller transfer function $C(s)$.**

   c. **If the reference $r(t)$ is a unit step, compute the plant input $u(t)$.** **(25 pts)**

a. The loop is internally stable since $T_o(s)$ is stable and the plant model is also stable and minimum phase (hence no unstable pole-zero cancellation can arise).

   Alternatively, C(s) can be calculated for part b. and the characteristic polynomial shown to have no roots with positive real parts.

b. Given $T_o(s)$ and $G_o(s)$ we can obtain $C(s)$ from:

$$C(s) = (G_o(s))^{-1} \frac{T_o(s)}{S_o(s)} = (G_o(s))^{-1} \frac{T_o(s)}{1 - T_o(s)} = \frac{4(s+1)}{s(s+4)}.$$

c. The plant input $u(t)$ can be computed from:

$$U(s) = S_{uo}(s)R(s) = \frac{T_o(s)}{G_o(s)}R(s) = \frac{4(s+1)}{s(s+2)^2} = \frac{1}{s} - \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

$$\Rightarrow u(t) = \mathcal{L}^{-1}[U(s)] = 1 - \mathrm{e}^{-2t} + 2t\,\mathrm{e}^{-2t}.$$

3. **In a nominal control loop, the sensitivity is given by:**

$$S_o(s) = \frac{s(s+4.2)}{s^2 + 4.2s + 9}.$$

Assume that the reference $r(t)$ is a unit step and that the output disturbance is given by $d_o(t) = 0.5\sin(0.2t)$.

Find an expression for the plant output $y(t)$ in the steady state. (25 pts)

We have that the output is given by:

$$Y(s) = T_o(s)R(s) + S_o(s)D_o(s).$$

In steady state there will be two components in $y(t)$. One, y_0, is forced by the constant reference, and is given by $y_0 = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = T_o(0)\times 1 = (1-S_o(0))\times 1 = 1$. The second component, $y_s(t)$, is forced by the disturbance sine-wave, and in steady state, this component can be computed using phasor analysis, i.e, by filtering the sine-wave (of frequency $\omega = 0.2$) by the frequency response of $S_o(s)$ at $s = j\omega = j0.2$. This yields:

$$y_s(t) = |S_o(j0.2)| \times 0.5\sin(0.2t + \angle S_o(j0.2)) = 0.0467\sin(0.2t + 1.5249).$$

Thus, the steady state plant output is given by $y(t) = y_0 + y_s(t) = 1 + 0.0467\sin(0.2t + 1.5249)$.

4. **Consider the following sets of plants and controllers with nominal models $G_o(s)$ and controllers $C(s)$. Assuming a one-degree-of-freedom unity control loop, use Routh's criterion to find the conditions for each of the controller's parameters under which the nominal feedback loop is stable.**

   a. **Nominal plant** $G_o(s) = \frac{1}{(s+1)^4}$**, with controller** $C(s) = K$**.**

   b. **Nominal plant** $G_o(s) = \frac{1}{(s+1)(s+2)}$**, with controller** $C(s) = \frac{as+b}{s}$**.** **(25 pts)**

a. For this plant and controller, the characteristic polynomial $p(s) = A_o(s)L(s) + B_o(s)P(s) = (s+1)^4 + K = s^4 + 4s^3 + 6s^2 + 4s + (1+K) = 0$, and consequently Routh's array is:

s^4	1	6	$1+K$
s^3	4	4	
s^2	5	$1+K$	
s^1	$\frac{16-4K}{5}$		
s^0	$1+K$		

For closed-loop stability, Routh's criterion states that there must be no changes of sign in the first column of the array. Consequently, $\frac{16-4K}{5} > 0$ and $K+1 > 0 \Rightarrow -1 < K < 4$.

b. For this plant and controller, the characteristic polynomial $p(s) = A_o(s)L(s) + B_o(s)P(s) = s(s+1)(s+2) + as + b = s^3 + 3s^2 + (a+2)s + b = 0$, and consequently Routh's array is:

s^3	1	$a+2$
s^2	3	b
s^1	$a+2-b/3$	
s^0	b	

To satisfy Routh's criterion, $b > 0$ and $a+2-b/3 > 0 \Rightarrow b > 0$ and $a > b/3-2$.