EE 4CL4 – Control System Design

Solutions to Homework Assignment #6

1. **For the system with the open-loop frequency response generating the Nyquist plot shown in Fig. 1, estimate the:**
   a. **stability gain margin,**
   b. **stability phase margin, and**
   c. **sensitivity peak.** **(25 pts)**

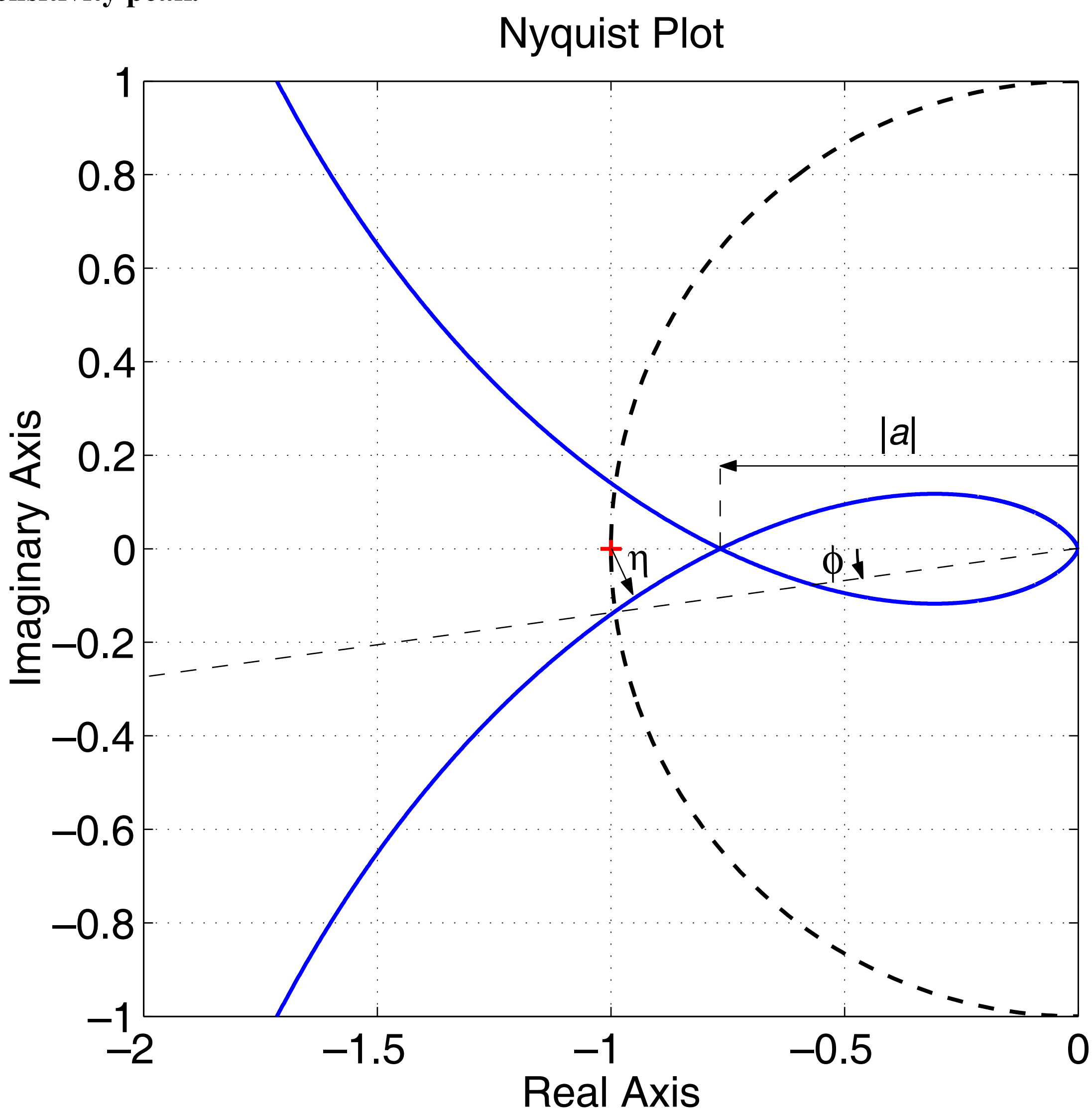

Figure 1

a. The stability gain margin $M_g \overset{\Delta}{=} -20\log_{10}(|a|)$. From Fig. 1, $a \approx -0.77 \Rightarrow M_g \approx 2.27$ dB.

b. The stability phase margin $M_f \overset{\Delta}{=} \phi$. From Fig. 1, $\phi \approx 0.135$ rad $\Rightarrow M_f \approx 0.135$ rad or 7.73°.

c. The sensitivity peak $\max_\omega S_o(j\omega) = 1/\eta$. From Fig. 1, $\eta \approx 0.12 \Rightarrow \max_\omega S_o(j\omega) \approx 8.33$.

2. **The nominal model for a plant is given by:**

$$G_o(s) = \frac{1}{(s+1)(-s+2)}.$$

Assume that this plant has to be controlled in a one-d.o.f. feedback loop such that the closed-loop characteristic polynomial is dominated by the factor $s^2 + 7s + 25$. Using the pole placement method, choose an appropriate minimum degree $A_{cl}(s)$ and synthesize a *biproper* controller $C(s)$ that has forced integration (i.e., one pole at $s = 0$). (25 pts)

We first notice that a minimum degree biproper controller (with integration) requires $A_{cl}(s)$ of degree 4 (= $2n$). We thus choose $A_{cl}(s) = (s^2 + 7s + 25)(s + 10)^2$. The choice of the double pole at $s = -10$ is arbitrary but for the requirement that they should generate modes faster than those produced by the factor $s^2 +7s+25$.

The associated Diophantine equation is:

$$\left(s^2 - s - 2\right)s\left(l_1 s + l_0\right) + (-1)\left(p_2 s^2 + p_1 s + p_0\right) = \left(s^2 + 7s + 25\right)(s+10)^2,$$

producing the pole-assignment matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_0 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 27 \\ 265 \\ 1200 \\ 2500 \end{bmatrix} \Rightarrow \begin{bmatrix} l_1 \\ l_0 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ -295 \\ -1256 \\ -2500 \end{bmatrix},$$

We thus obtain:

$$C(s) = \frac{-\left(295s^2 + 1256s + 2500\right)}{s(s+28)}.$$

3. **Consider the feedback control of an unstable plant. Prove that the controller output $u(t)$, exhibits undershoot for any step reference and for any step-output disturbance. (25 pts)**

The transfer function from the reference and the output disturbance to the controller output is given by:

$$U(s) = S_{uo}(s)(R(s) - D_o(s))$$

When the plant has unstable poles, they cannot be cancelled and thus they appear as non-minimum-phase zeros in $S_{uo}(s)$. If $z_o > 0$ is any non-minimum-phase zero in $S_{uo}(s)$, then $S_{uo}(z_o) = 0$. From Lemma 4.1 on Page 81 of Goodwin et al.:

$$\int_0^\infty u(t)e^{-z_0 t}\,\mathrm{d}t = \lim_{s \to z_o} U(s) = 0.$$

For this equation to be satisfied, $u(t)$ will necessarily exhibit undershoot (i.e., be negative for some period of the response) for any step reference and step-output disturbance.

4. **The nominal model for a plant is given by:**

$$G_o(s) = \frac{5(s-1)}{(s+1)(s-5)}.$$

This plant has to be controlled in a one-d.o.f. unity-feedback loop.

a. **Determine the time-domain *integral constraints* for the plant input $u(t)$, the plant output y(t), and the controller error $e(t)$ in the loop. Assume exact inversion at $\omega = 0$ (see page 210 of Goodwin et al.) and step-like reference and disturbances (input and output).**

b. **Discuss why the control of this nominal plant especially difficult. Hint: What constraints should be placed on the closed-loop bandwidth? (25 pts)**

a. In this particular case we have that the plant model and the controller satisfies $B_o(1) = 0$; $A_o(5) = 0$; $L(0) = 0$. The zero in $L(s)$ at $s = 0$ is required for exact inversion at $\omega = 0$.

The constraints for the sensitivities derive from the interpolation constraints required to achieve internal stability (no cancellation of unstable poles and NMP zeros). These constraints are:

$S_o(1) = 1$; $S_o(5) = 0$; $T_o(1) = 0$; $T_o(5) = 1$; $S_{io}(1) = 0$; $S_{uo}(5) = 0$.

First, the reference effect will be seen in $y(t)$, $u(t)$ and $e(t)$, and their integral properties:

$$Y(s) = T_o(s)\frac{1}{s} \Rightarrow \int_0^\infty y(t)\mathrm{e}^{-t}\mathrm{d}\,t = 0$$

$$\Rightarrow \int_0^\infty y(t)\mathrm{e}^{-5t}\mathrm{d}\,t = \frac{1}{5}$$

$$U(s) = S_{uo}(s)\frac{1}{s} \Rightarrow \int_0^\infty u(t)\mathrm{e}^{-5t}\mathrm{d}\,t = 0$$

$$E(s) = S_o(s)\frac{1}{s} \Rightarrow \int_0^\infty e(t)\mathrm{e}^{-5t}\mathrm{d}\,t = 0$$

$$\Rightarrow \int_0^\infty e(t)\mathrm{e}^{-t}\mathrm{d}\,t = 1$$

Likewise, the effect of a unit step *input* disturbance is:

$$Y(s) = S_{io}(s)\frac{1}{s} \Rightarrow \int_0^\infty y(t)\mathrm{e}^{-t}\mathrm{d}\,t = 0$$

$$U(s) = -T_o(s)\frac{1}{s} \Rightarrow \int_0^\infty u(t)\mathrm{e}^{-t}\mathrm{d}\,t = 0$$

$$\Rightarrow \int_0^\infty u(t)\mathrm{e}^{-5t}\mathrm{d}\,t = -\frac{1}{5}$$

$$E(s) = -S_{io}(s)\frac{1}{s} \Rightarrow \int_0^\infty e(t)\mathrm{e}^{-t}\mathrm{d}\,t = 0$$

The effect of a unit step *output* disturbance is:

$$Y(s) = S_o(s)\frac{1}{s} \Rightarrow \int_0^\infty y(t)\mathrm{e}^{-5t}\mathrm{d}\,t = 0$$

$$\Rightarrow \int_0^\infty y(t)\mathrm{e}^{-t}\mathrm{d}\,t = 1$$

$$U(s) = -S_{uo}(s)\frac{1}{s} \Rightarrow \int_0^\infty u(t)\mathrm{e}^{-5t}\mathrm{d}\,t = 0$$

$$E(s) = -S_o(s)\frac{1}{s} \Rightarrow \int_0^\infty e(t)\mathrm{e}^{-5t}\mathrm{d}\,t = 0$$

$$\Rightarrow \int_0^\infty e(t)\mathrm{e}^{-t}\mathrm{d}\,t = -1$$

b. This case is especially difficult because of contradicting requirements:

- The open-loop NMP zero at s = 1 sets an *upper bound* for the closed-loop bandwidth, since the integral constraint $\int_0^\infty y(t)\mathrm{e}^{-t}\mathrm{d}\,t = 0$ derived above says that a plant output which settles much faster than e^{-t} will exhibit a *large undershoot*.
- The unstable open-loop pole at s = 5 sets a *lower bound* for the closed loop bandwidth, since the integral constraint $\int_0^\infty e(t)\mathrm{e}^{-5t}\mathrm{d}\,t = 0$ derived above says that a plant output which settles much slower than e^{-5t} will exhibit a *large overshoot*.