EE 4CL4 – Control System Design

Solutions to Homework Assignment #7

1. **Consider a plant with the nominal model:**

$$G_o(s) = \frac{e^{-0.5s}(s+5)}{(s+1)(s+3)}.$$

Build a Smith predictor such that the dominant closed-loop poles are located at s = –2 ± *j*0.5 and the controller *C*(*s*) has forced integration (i.e., one pole at *s* = 0).

What is the nominal complementary sensitivity of the closed-loop system? (25 pts)

A Smith predictor is shown in Figure 7.1 of Goodwin et al. We need to synthesize a controller considering only the rational part, $\overline{G}_o(s)$, of the nominal model, $G_o(s)$, where:

$$\overline{G}_o(s) = \frac{s+5}{(s+1)(s+3)}.$$

If the dominant closed loop poles are $-2 \pm j0.5$, and we require integration, we can build a closed loop polynomial of the form $A_{cl}(s) = (s^2+4s+4.25)(s^2+8s+16)$. Thus:

$$(s+1)(s+3)s(l_1 s + l_0) + (s+5)(p_2 s^2 + p_1 s + p_0) = (s^2+4s+4.25)(s^2+8s+16),$$

producing the pole-assignment matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 \\ 3 & 4 & 5 & 1 & 0 \\ 0 & 3 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} l_1 \\ l_0 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 52.25 \\ 98 \\ 68 \end{bmatrix} \Rightarrow \begin{bmatrix} l_1 \\ l_0 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4.7687 \\ 3.2313 \\ 14.0188 \\ 13.6 \end{bmatrix}.$$

Consequently the controller has the denominator and numerator polynomials $L(s) = s(s + 4.7687)$ and $P(s) = 3.2313s^2 + 14.0188s + 13.6$, respectively.

The nominal complementary sensitivity is then:

$$T_o(s) = \frac{e^{-0.5s}\, C(s)\overline{G}_o(s)}{1 + C(s)\overline{G}_o(s)} = \frac{e^{-0.5s}\, 3.2313(s+5)(s+2.874)(s+1.464)}{(s^2+4s+4.25)(s^2+8s+16)}.$$

2. **The nominal model for a plant is given by:**

$$G_o(s) = \frac{1}{(s+2)(s+1)^2}.$$

Using the Ziegler-Nichols Oscillation Method to determine the controller parameters, design a PI controller *that prevents wind-up* if the plant actuator is known to have a maximal movement limit for an input $u(t)$ of −3 and +5. (25 pts)

Under proportional control with gain K_p, the closed-loop characteristic polynomial $A_{cl}(s) = s^3+4s^2+5s+2+K_p$. At the critical-stability point, $K_p = K_c$ and $s = j\omega_c$, such that:

$$-j\omega_c^3 - 4\omega_c^2 + 5j\omega_c + 2 + K_c = 0 \Rightarrow K_c = j\omega_c^3 + 4\omega_c^2 - 5j\omega_c - 2.$$

The critical gain $K_c \in \Re$, so the complex term in the equation above must equal zero, which gives:

$$\omega_c^{\,3} - 5\omega_c = 0 \Rightarrow \omega_c = \sqrt{5} \Rightarrow P_c = \frac{2\pi}{\omega_c} = \frac{2\pi}{\sqrt{5}}.$$

Substituting the value for ω_c into the equation for K_c yields:

$$K_c = 4\omega_c^{\,2} - 2 = 4 \cdot 5 - 2 = 18.$$

From Table 6.1 of Goodwin et al., the PI parameters are then:

$$K_p = 0.45K_c = 8.1, \text{ and}$$

$$T_r = \frac{P_c}{1.2} = 2.3416,$$

producing the PI controller:

$$C_{PI}(s) = 8.1\left(1 + \frac{1}{2.3416s}\right) = \frac{8.1s + 3.4592}{s}.$$

It is possible to implement this PI controller in the architectural form shown in Fig. 8.9 of Goodwin et al. by setting $p_1 = 8.1$ and $p_0 = 3.4592$. To avoid wind-up, the limiter needs to be of the form shown in Fig. 8.10 of Goodwin et al. with $u_{\min} = -3$ and $u_{\max} = 5$.

3. **Determine the steady-state error in response to an "acceleration" reference $r(t) = At^2/2$ for a one-d.o.f. unity-feedback control loop with:**
   a. **one,**
   b. **two, and**
   c. **three**

   open-loop integrators (i.e., poles at $s = 0$), assuming that the open-loop controller and plant satisfy Eqs. (8.6.8)–(8.6.10) of Goodwin et al. (25 pts)

   The Laplace transform of the "acceleration" reference $r(t)$ is:

$$R(S) = A/s^3 \ ,$$

   producing the steady state error for this system:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sR(S)}{1+G_o(S)C(S)} = \lim_{s\to 0} \frac{A}{s^2 G_o(S)C(S)} .$$

   a. If the control loop has only one open-loop integrator, then an s term remains in the denominator of the equation above and the steady-state error is therefore infinite.

   b. If the control loop has two open-loop integrators, then no s term remains in the denominator of the equation above and the steady-state error is therefore Ac_0/c_1.

   c. If the control loop has three open-loop integrators, then there is an s term in the numerator of the equation above and the steady-state error is therefore zero.

4. **Consider the feedback control of a plant with nominal model:**

$$G_o(s) = \frac{s+1}{s(s+3)} .$$

   a. **Use the pole placement method to synthesize a controller such that the closed-loop poles are at (–2; –2; –2).**
   b. **Prove that a one-d.o.f. unity-feedback control loop consisting of this plant and controller must exhibit overshoot in response to a step reference.**
   c. **Design a reference prefilter $H(s)$ to create a two-d.o.f. control loop that does not exhibit overshoot. (25 pts)**

a. The close-loop characteristic polynomial is $A_{cl}(s) = s^3+6s^2+12s+8$. Thus:

$$s(s+3)(l_1 s + l_0) + (s+1)(p_1 s + p_0) = s^3 + 6s^2 + 12s + 8,$$

producing the pole-assignment matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 12 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 2.5 \\ 8 \end{bmatrix},$$

giving the controller $C(s) = \dfrac{5s+16}{2s+1}$.

b. The nominal complimentary sensitivity of the controller and nominal plant model above placed in a one-d.o.f. unity-feedback loop is:

$$T_o(s) = \frac{2.5(s+1)(s+3.2)}{(s+2)^3}.$$

Consequently, $T_o(-1) = 0$ and $S_o(-1) = 1$. The error in response to a unit step reference is then:

$$E(s) = S_o(s)\frac{1}{s} \Rightarrow \int_0^\infty e(t)\mathrm{e}^t \mathrm{d}\,t = \lim_{s\to -1} S_o(s)\frac{1}{s} = -1.$$

The error is initially positive, but the integral is negative, thus the error must change sign, i.e., the step response exhibits overshoot.

An alternative solution is to calculate the step response $y(t)$ and show that $y(t) > y_\infty$ for some value of t between 0 and ∞.

c. In order to remove the overshoot, the reference prefilter $H(s)$ must cancel the plant zero to the right of the closed-loop poles, i.e., at $s = -1$. Choosing:

$$H(s) = \frac{1}{s+1}$$

produces a step response:

$$Y(s) = T_o(s)H(s)R(s) = \frac{2.5(s+1)(s+3.2)}{(s+2)^3}\frac{1}{s+1}\frac{1}{s} = \frac{2.5(s+3.2)}{s(s+2)^3}$$

$$\Rightarrow y(t) = L^{-1}[Y(s)] = 1 - \frac{3}{4}t^2\,\mathrm{e}^{-2t} - 2t\,\mathrm{e}^{-2t} - \mathrm{e}^{-2t},$$

We find that $y(t) < y_\infty$ for $0 < t < \infty$, i.e., the step response exhibits no overshoot.