ELEC ENG 4CL4 – Control System Design

Solutions to Homework Assignment #1

1. Consider a system that obeys the differential equation:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0.$$

a. Linearize this equation around the operating point $x = \pi/4$.

b. **Derive a state-space representation of the linear equation found in part a. (25 pts)**

a. The nonlinear term $\cos x$ is linearized via the Taylor series approximation:

$$f(x) = \cos x \approx \cos\left(\frac{\pi}{4}\right) + \left.\frac{df}{dx}\right|_{x=\frac{\pi}{4}} \left(x - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} - \frac{x}{\sqrt{2}},$$

giving the linear equation:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - \frac{x}{\sqrt{2}} = -\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}.$$

Note that an equivalent model in terms of $\Delta x = x - \frac{\pi}{4}$ is:

$$\frac{d^2\Delta x}{dt^2} + 2\frac{d\Delta x}{dt} - \frac{\Delta x}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

b. To obtain a state-space representation of the linearized model, we define two state variables $x_1 = x$ and $x_2 = \frac{dx}{dt}$, the input variable $u = \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$, and the output variable $y = x$, giving:

$$\frac{dx_1}{dt} = x_2,$$
$$\frac{dx_2}{dt} = \frac{x_1}{\sqrt{2}} - 2x_2 - u,$$
$$y = x_1.$$

Note that an equivalent model in terms of $\Delta x = x - \frac{\pi}{4}$ and $\Delta u = \frac{1}{\sqrt{2}}$ is:

$$\frac{d\Delta x_1}{dt} = \Delta x_2,$$
$$\frac{d\Delta x_2}{dt} = \frac{\Delta x_1}{\sqrt{2}} - 2\Delta x_2 - \Delta u,$$
$$\Delta y = \Delta x_1.$$

2. A two-phase (i.e., two-input) permanent magnet stepper motor can be described by the following set of differential equations:

$$\frac{d^2\theta}{dt^2} = -K_2 i_a \sin\left(K_3\theta\right) + K_2 i_b \cos\left(K_3\theta\right) - K_1\frac{d\theta}{dt},$$
$$\frac{di_a}{dt} = -K_5 i_a + K_4\frac{d\theta}{dt}\sin\left(K_3\theta\right) + K_6 v_a,$$
$$\frac{di_b}{dt} = -K_5 i_b - K_4\frac{d\theta}{dt}\cos\left(K_3\theta\right) + K_6 v_b,$$

where θ is angular displacement of the rotor, i_a and i_b are the currents in the two phases, v_a and v_b are the voltages applied the two phases (i.e., the inputs), and $K_1,\ldots,K_6$ are constants.

a. Derive a state-space representation of this system.

b. **Linearize the state-space model found in part a around the operating point** $\theta = \text{constant}$**.**

(25 pts)

a. We assign four state variables $x_1 = \theta$, $x_2 = \frac{d\theta}{dt}$, $x_3 = i_a$ and $x_4 = i_b$, two input variables $u_1 = v_a$ and $u_2 = v_b$, and one output variable $y = \theta$, giving the state-space representation:

$$\frac{dx_1}{dt} = f_1\left(x_1,x_2,x_3,x_4,u_1,u_2\right) = x_2$$
$$\frac{dx_2}{dt} = f_2\left(x_1,x_2,x_3,x_4,u_1,u_2\right) = -K_2x_3\sin\left(K_3x_1\right) + K_2x_4\cos\left(K_3x_1\right) - K_1x_2,$$
$$\frac{dx_3}{dt} = f_3\left(x_1,x_2,x_3,x_4,u_1,u_2\right) = -K_5x_3 + K_4x_2\sin\left(K_3x_1\right) + K_6u_1,$$
$$\frac{dx_4}{dt} = f_4\left(x_1,x_2,x_3,x_4,u_1,u_2\right) = -K_5x_4 - K_4x_2\cos\left(K_3x_1\right) + K_6u_2,$$
$$y = g\left(x_1,x_2,x_3,x_4,u_1,u_2\right) = x_1.$$

b. To simplify the linearization of this state-space model, we reformulate it in terms of the distances from the operating point $\left(x_{1Q},x_{2Q},x_{3Q},x_{4Q},u_{1Q},u_{2Q},y_Q\right)$:

$$\Delta x_1 = x_1 - x_{1Q},$$
$$\Delta x_2 = x_2 - x_{2Q},$$
$$\Delta x_3 = x_3 - x_{3Q},$$
$$\Delta x_4 = x_4 - x_{4Q},$$
$$\Delta u_1 = u_1 - u_{1Q},$$
$$\Delta u_2 = u_2 - u_{2Q},$$
$$\Delta y = y - y_Q.$$

Defining the vectors:

$$\mathbf{\Delta x} \triangleq \left[\Delta x_1 \quad \Delta x_2 \quad \Delta x_3 \quad \Delta x_4\right]^{\mathrm{T}},\ \mathbf{\Delta \dot{x}} \triangleq \left[\Delta \dot{x}_1 \quad \Delta \dot{x}_2 \quad \Delta \dot{x}_3 \quad \Delta \dot{x}_4\right]^{\mathrm{T}} \text{ and } \mathbf{\Delta u} \triangleq \left[\Delta u_1 \quad \Delta u_2\right]^{\mathrm{T}},$$

The linearized state-space model is obtained from the Taylor series approximation in the matrix formulation:

$$\begin{aligned}
\mathbf{\Delta \dot{x}} &= \begin{bmatrix}
\left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_1}{\partial x_3}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_1}{\partial x_4}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} \\
\left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_2}{\partial x_3}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_2}{\partial x_4}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} \\
\left.\frac{\partial f_3}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_3}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_3}{\partial x_3}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_3}{\partial x_4}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} \\
\left.\frac{\partial f_4}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_4}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_4}{\partial x_3}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_4}{\partial x_4}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}}
\end{bmatrix} \mathbf{\Delta x} + \begin{bmatrix}
\left.\frac{\partial f_1}{\partial u_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_1}{\partial u_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} \\
\left.\frac{\partial f_2}{\partial u_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_2}{\partial u_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} \\
\left.\frac{\partial f_3}{\partial u_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_3}{\partial u_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} \\
\left.\frac{\partial f_4}{\partial u_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial f_4}{\partial u_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}}
\end{bmatrix} \mathbf{\Delta u} \\
&= \begin{bmatrix}
0 & 1 & 0 & 0 \\
-K_2 K_3 x_{3Q} \cos\left(K_3 x_{1Q}\right) - K_2 K_3 x_{4Q} \sin\left(K_3 x_{1Q}\right) & -K_1 & -K_2 \sin\left(K_3 x_{1Q}\right) & K_2 \cos\left(K_3 x_{1Q}\right) \\
K_3 K_4 x_{2Q} \cos\left(K_3 x_{1Q}\right) & K_4 \sin\left(K_3 x_{1Q}\right) & -K_5 & 0 \\
K_3 K_4 x_{2Q} \sin\left(K_3 x_{1Q}\right) & -K_4 \cos\left(K_3 x_{1Q}\right) & 0 & -K_5
\end{bmatrix} \mathbf{\Delta x} \\
&\quad + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ K_6 & 0 \\ 0 & K_6 \end{bmatrix} \mathbf{\Delta u},
\end{aligned}$$

$$\begin{aligned}
\Delta y &= \begin{bmatrix}
\left.\frac{\partial g}{\partial x_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial g}{\partial x_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial g}{\partial x_3}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial g}{\partial x_4}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}}
\end{bmatrix} \mathbf{\Delta x} + \begin{bmatrix}
\left.\frac{\partial g}{\partial u_1}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}} & \left.\frac{\partial g}{\partial u_2}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}}
\end{bmatrix} \mathbf{\Delta u} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{\Delta x} + \begin{bmatrix} 0 & 0 \end{bmatrix} \mathbf{\Delta u}.
\end{aligned}$$

3. Given the following differential equation, solve for $y(t)$ using the Laplace transform if all initial conditions are zero:

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32\mu(t),$$

where $\mu(t)$ is the unit step function. **(25 pts)**

The Laplace transform of this equation is:

$$s^2Y(s) + 12sY(s) + 32Y(s) = 32\frac{1}{s}$$

$$\Rightarrow Y(s)\left\{s^2 + 12s + 32\right\} = \frac{32}{s}.$$

Solving for $Y(s)$ gives:

$$\begin{aligned} Y(s) &= \frac{32}{s\left(s^2 + 12s + 32\right)} \\ &= \frac{32}{s(s+4)(s+8)} \\ &= \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}. \end{aligned}$$

Taking the inverse Laplace transform of $Y(s)$ we obtain:

$$y(t) = 1 - 2e^{-4t} + e^{-8t}, \quad t \geq 0.$$

4. A system has the transfer function:

$$H(s) = \frac{10}{(s+7)(s+8)}.$$

a. Compute the system's response to the Dirac delta function (unit impulse) $\delta_{\mathrm{D}}(t)$.

b. **Compute the system's response to the unit step function $\mu(t)$.** **(25 pts)**

a. Because the Laplace transform of $\delta_{\mathrm{D}}(t)$ is 1, the impulse response of a system is the inverse Laplace transform of its transfer function:

$$h(t) = \mathfrak{L}^{-1}\{H(s)\} = \mathfrak{L}^{-1}\left\{\frac{10}{(s+7)(s+8)}\right\} = \mathfrak{L}^{-1}\left\{\frac{10}{s+7} - \frac{10}{s+8}\right\}$$
$$= 10e^{-7t} - 10e^{-8t}, \quad t \geq 0.$$

b. The step response of this system is:

$$y(t) = \mathfrak{L}^{-1}\{H(s)U(s)\} = \mathfrak{L}^{-1}\left\{\frac{10}{(s+7)(s+8)}\frac{1}{s}\right\} = \mathfrak{L}^{-1}\left\{\frac{\frac{10}{56}}{s} - \frac{\frac{80}{56}}{s+7} + \frac{\frac{5}{4}}{s+8}\right\}$$
$$= \tfrac{10}{56} - \tfrac{80}{56}e^{-7t} + \tfrac{5}{4}e^{-8t}, \quad t \geq 0.$$