ELEC ENG 4CL4 – Control System Design

Solutions to Homework Assignment #2

1. A system has the transfer function:

$$H(s) = \frac{s + \frac{1}{2}}{s^2 + 2s + 1}.$$

a. **Will the step response of this system exhibit undershoot or overshoot? Explain your answer in terms of the locations of the poles and zeros of the transfer function.**

b. **If the step response does exhibit undershoot or overshoot, find both the *magnitude* and the *time* of maximum undershoot or overshoot.** **(25 pts)**

a. This transfer function has two real-valued poles at $s = -1$, which will not contribute directly to overshoot or undershoot. (A pair of complex conjugate poles could contribute to overshoot.) The transfer function has one minimum-phase zero at $s = -\frac{1}{2}$. *Minimum-phase* zeros can produce overshoot; conversely, *nonminimum-phase* zeros can produce undershoot. The closer the zero is to the imaginary axis, i.e., the smaller the magnitude of the zero, the larger the contribution to overshoot or undershoot. We note that the minimum-phase zero at $s = -\frac{1}{2}$ is closer to the imaginary axis than the poles at $s = -1$, so we would expect that the system would exhibit *overshoot*.

b. The Laplace transform of the step response $y(t)$ of this system is:

$$Y(s) = H(s)U(s) = H(s)\frac{1}{s} = \frac{s + \frac{1}{2}}{s(s+1)^2} = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{(s+1)^2}.$$

Taking the inverse Laplace transform of $Y(s)$ gives:

$$y(t) = \tfrac{1}{2} - \tfrac{1}{2}e^{-t} + \tfrac{1}{2}te^{-t}, \quad t \geq 0.$$

We find the maximum of $y(t)$ by setting slope of $y(t)$ equal to zero:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-t} - \tfrac{1}{2}te^{-t} = 0.$$

One solutions is at $t \to \infty$, which corresponds to the steady-state response $y_\infty = \frac{1}{2}$, not the maximum. The second solution, which corresponds to the maximum, can be found as follows:

$$\begin{aligned}
&\Rightarrow \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-t} - \tfrac{1}{2}te^{-t} = 0 \\
&\Rightarrow e^{-t} = \tfrac{1}{2}te^{-t} \\
&\Rightarrow 1 = \tfrac{1}{2}t \\
&\Rightarrow t = 2,
\end{aligned}$$

giving a maximum value of $y(2) = \frac{1}{2} - \frac{1}{2}e^{-2} + \frac{1}{2}\cdot 2 \cdot e^{-2} \approx 0.5677$.

Consequently, the overshoot is $M_p = y(2) - y_\infty = 0.0667$, occurring at $t = 2$.

2. A one-d.o.f., unity-feedback control loop has the following controller and plant transfer functions:

$$C(s) = \frac{2(s+1)}{s} \qquad \textbf{and} \qquad G_o(s) = \frac{s+3}{(s+2)(s+4)}.$$

a. **Use the Final Value Theorem (given in the table of Laplace transform properties) to calculate the *steady-state* plant output y_∞ in response to a unit step change in the reference, i.e., $r(t) = \mu(t)$.**

b. **Find the steady-state plant output $y(t)$ for the output disturbance $d_o(t) = 0.5\sin(2t)$. (25 pts)**

a. In a one-d.o.f., unity-feedback nominal control loop, the plant output is related to the reference signal according to the nominal complementary sensitivity:

$$Y(s) = T_o(s)R(s) = T_o(s)\frac{1}{s}.$$

For the given control and plant transfer functions, the nominal complementary sensitivity is:

$$T_o(s) = \frac{G_o(s)C(s)}{1+G_o(s)C(s)} = \frac{B_o(s)P(s)}{A_o(s)L(s)+B_o(s)P(s)} = \frac{2(s+1)(s+3)}{s(s+2)(s+4)+2(s+1)(s+3)}.$$

The final value theorem gives that the steady-state plant output is:

$$\begin{aligned}
y_\infty &= \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sT_o(s)\frac{1}{s} \\
&= \frac{2(0+1)(0+3)}{0(0+2)(0+4)+2(0+1)(0+3)} \\
&= \frac{6}{0+6} \\
&= 1.
\end{aligned}$$

b. In a one-d.o.f., unity-feedback nominal control loop, the plant output is related to an output disturbance according to the nominal sensitivity:

$$Y(s) = S_o(s) D_o(s).$$

For the given control and plant transfer functions, the nominal sensitivity is:

$$S_o(s) = \frac{1}{1+G_o(s)C(s)} = \frac{A_o(s)L(s)}{A_o(s)L(s)+B_o(s)P(s)}$$
$$= \frac{s(s+2)(s+4)}{s(s+2)(s+4)+2(s+1)(s+3)} = \frac{s^3+6s^2+8s}{s^3+8s^2+16s+6}.$$

For a sinusoidal disturbance, the steady-state response can be found by evaluating the frequency response of the nominal sensitivity:

$$S_o(j\omega) = \frac{-j\omega^3-6\omega^2+8j\omega}{-j\omega^3-8\omega^2+16j\omega+6},$$

at the frequency of the sinusoid $\omega = 2$:

$$S_o(j2) = \frac{-j2^3-6\cdot 2^2+8j2}{-j2^3-8\cdot 2^2+16j2+6}$$
$$= \frac{-24+j8}{-26+j24}$$
$$= 0.6518 \ + \ j0.2939,$$

giving:

$$y(t) = \left|S_o(j2)\right| 0.5\sin\left(2t+\angle S_o(j2)\right)$$
$$= 0.7150\cdot 0.5\sin(2t+0.4237)$$
$$= 0.3575\sin(2t+0.4237).$$

3. Find the range of values of K for which the controller:

$$C(s) = \frac{K(s+3)}{s+2}$$

stabilizes the unstable plant model:

$$G_o(s) = \frac{2}{(s+4)(s-1)}$$

when placed together in a one-d.o.f., unity-feedback control loop. (25 pts)

This control loop has the characteristic polynomial:

$$\begin{aligned} A_{cl}(s) &= A_o(s)L(s) + B_o(s)P(s) \\ &= (s+2)(s+4)(s-1) + 2K(s+3) \\ &= s^3 + 5s^2 + (2+2K)s - 8 + 6K. \end{aligned}$$

Evaluating Routh's array:

$$\begin{array}{lll} s^3 & 1 & 2+2K \\ s^2 & 5 & -8+6K \\ s^1 & \dfrac{4K+18}{5} & \\ s^0 & 6K-8 & \end{array}$$

gives two conditions for stability:

$$\frac{4K+18}{5} > 0 \Rightarrow K > -\frac{18}{4} \qquad \text{and} \qquad 6K - 8 > 0 \Rightarrow K > \frac{8}{6}.$$

We note that satisfying the second criterion also satisfies the first, so K must be greater than $8/6$ to stabilize this control loop.

4. Find the additive modeling error (AME) $G_\epsilon(s)$ and the multiplicative modeling error (MME) $G_\Delta(s)$ when the calibration and nominal plant models are given by:

$$G(s) = \frac{-\frac{1}{2}s + 1}{(s+1)^2} \qquad \textbf{and} \qquad G_o(s) = \frac{1}{(s+1)^2},$$

respectively

Is the AME low-pass, band-pass or high-pass? What about the MME? (25 pts)

The AME and MME are:

$$G_\epsilon(s) = G(s) - G_o(s) = \frac{-\frac{1}{2}s}{(s+1)^2} \qquad \text{and} \qquad G_\Delta(s) = \frac{G(s)}{G_o(s)} - 1 = -\tfrac{1}{2}s,$$

respectively, and their frequency responses are:

$$G_\epsilon(j\omega) = \frac{-\frac{1}{2}j\omega}{(j\omega+1)^2} \qquad \text{and} \qquad G_\Delta(j\omega) = -\tfrac{1}{2}j\omega,$$

which are band-pass and high-pass, respectively.