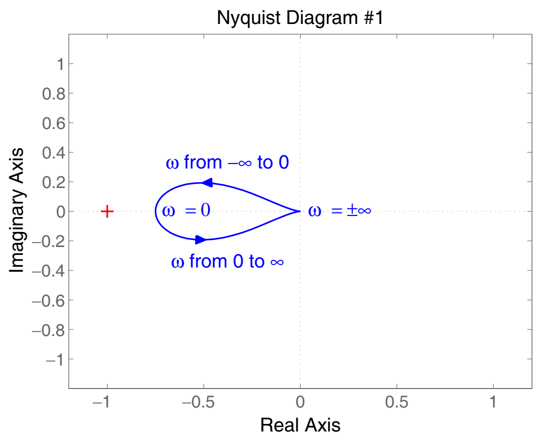
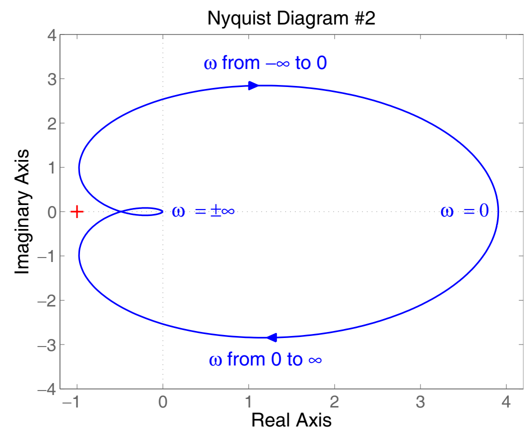
ELEC ENG 4CL4 – Control System Design

Solutions to Homework Assignment #3

1. Consider the following two Nyquist diagrams labelled #1 and #2:





For each of these Nyquist diagrams, what can you say about:

a. the *open-loop stability* of the plant and controller, and

b. the *closed-loop stability* of the plant and controller? (20 pts)

a. It is *not* possible to known the open-loop stability directly from the Nyquist plots. However, Nyquist diagrams #1 and #2 are typical of systems that have one unstable open-loop pole and no unstable open-loop poles, respectively.

b. Assuming the open-loop stability of each system given in part a., the system in diagram #1 has $P = 1$ unstable open-loop pole and $N = 0$ encirclements of the point $(-1,0)$ on the Nyquist plane. The number of closed-loop unstable poles is:

$$Z = N + P = 0 + 1 = 1,$$

and consequently the system in diagram #1 is also *closed-loop unstable*.

The system in diagram #2 has $P = 0$ unstable open-loop pole and $N = 0$ encirclements of the point $(-1,0)$ on the Nyquist plane. The number of closed-loop unstable poles is:

$$Z = N + P = 0 + 0 = 0,$$

and consequently the system in diagram #2 is also *closed-loop stable*.

2. **Determine the PID controller parameters (for the *standard form*) for a plant with the nominal model:**

$$G_o(s) = \frac{2}{(s+2)(s+1)^2},$$

using the Ziegler-Nichols oscillation method. (30 pts)

The closed-closed characteristic polynomial for this nominal plant model in a one-d.o.f. unity-feedback loop with a proportional controller is:

$$1 + K_p G_o(s) = (s+2)(s+1)^2 + 2K_p = 0.$$

At the point of critical stability, $K_p = K_c$ and $s = j\omega_c$, such that:

$$(s+2)(s+1)^2 + 2K_p = (j\omega_c + 2)(j\omega_c + 1)^2 + 2K_c = 0$$

$$\Rightarrow K_c = \tfrac{1}{2} j\omega_c^3 + 2\omega_c^2 - \tfrac{5}{2} j\omega_c - 1 = 2\omega_c^2 - 1 + j\left(\tfrac{1}{2}\omega_c^3 - \tfrac{5}{2}\omega_c\right).$$

The critical gain $K_c \in \Re$, so the imaginary term in the equation above must equal zero, which gives:

$$\tfrac{1}{2}\omega_c^3 - \tfrac{5}{2}\omega_c = 0 \Rightarrow \omega_c = \sqrt{5} \Rightarrow P_c = \frac{2\pi}{\omega_c} = \frac{2\pi}{\sqrt{5}}.$$

Substituting the value for ω_c into the equation for K_c yields:

$$K_c = 2\omega_c^2 - 1 = 2 \cdot 5 - 1 = 9.$$

From Table 6.1 of Goodwin et al., the PID parameters are then:

$$K_p = 0.6K_c = 5.4,$$

$$T_r = 0.5P_c = 1.4050, \text{ and}$$

$$T_d = P_c/8 = 0.3512.$$

3. Find suitable PID controller parameters (for the *standard form*) for a plant with the nominal model:

$$G_o(s) = \frac{10}{(s+2)(s+5)}, \qquad (1)$$

using the reaction curve method with:

a. the Ziegler-Nichols parameters, and

b. the Cohen-Coon parameters. **(30 pts)**

The process reaction curve for this plant can be obtained by calculating the unit step response $y(t)$ of the plant in open loop:

$$Y(s) = G_o(s)U(s) = \frac{10}{s(s+2)(s+5)}$$

$$\Rightarrow y(t) = \mathfrak{L}^{-1}\left[Y(s)\right] = \mathfrak{L}^{-1}\left[\frac{10}{s(s+2)(s+5)}\right] = 1 - \frac{5}{3}\mathrm{e}^{-2t} + \frac{2}{3}\mathrm{e}^{-5t}.$$

The slope of $y(t)$ is then:

$$\dot{y}(t) = \frac{10}{3}\mathrm{e}^{-2t} - \frac{10}{3}\mathrm{e}^{-5t},$$

and the maximal slope can be found at the time t when the derivate of the slope is zero:

$$\ddot{y}(t) = -\frac{20}{3}\mathrm{e}^{-2t} + \frac{50}{3}\mathrm{e}^{-5t} = 0$$

$$\Rightarrow -20\mathrm{e}^{-2t} + 50\mathrm{e}^{-5t} = 0$$

$$5\mathrm{e}^{-5t} = 2\mathrm{e}^{-2t}$$

$$5\mathrm{e}^{-5t}\mathrm{e}^{2t} = 2$$

$$\mathrm{e}^{-3t} = 2/5$$

$$-3t = \log_{\mathrm{e}}(2/5)$$

$$t = -\frac{\log_{\mathrm{e}}(2/5)}{3} \approx 0.3054$$

The maximal slope is then:

$$\dot{y}(0.3054) = \frac{10}{3}\mathrm{e}^{-2\cdot 0.3054} - \frac{10}{3}\mathrm{e}^{-5\cdot 0.3054} = 1.0858,$$

at the point $t = 0.3054$, $y(0.3054) = 0.2399$, giving the maximum slope tangent:

$$m.s.t. = 1.0858(t - 0.3054) + 0.2399.$$

The *m.s.t.* is equal to $y_0 = 0$ at time $t_1 = 0.0845$ and is equal to $y_\infty = 1$ at $t_2 = 1.0054$. The unit step ($u_0 = 0$; $u_\infty = 1$) was applied at time $t_0 = 0$, giving the parameter model:

$$K_0 = \frac{y_\infty - y_0}{u_\infty - u_0} = 1; \qquad \tau_0 = t_1 - t_0 = 0.0845; \qquad \nu_0 = t_2 - t_1 = 0.9210.$$

a. From Table 6.2 of Goodwin et al., the *Ziegler-Nichols* PID parameters are then:

$$K_p = \frac{1.2\nu_0}{K_0\tau_0} = 13.0793,$$

$$T_r = 2\tau_0 = 0.1690, \text{ and}$$

$$T_d = 0.5\tau_0 = 0.0423.$$

b. From Table 6.3 of Goodwin et al., the *Cohen-Coon* PID parameters are:

$$K_p = \frac{\nu_0}{K_0\tau_0}\left[\frac{4}{3} + \frac{\tau_0}{4\nu_0}\right] = 14.7825,$$

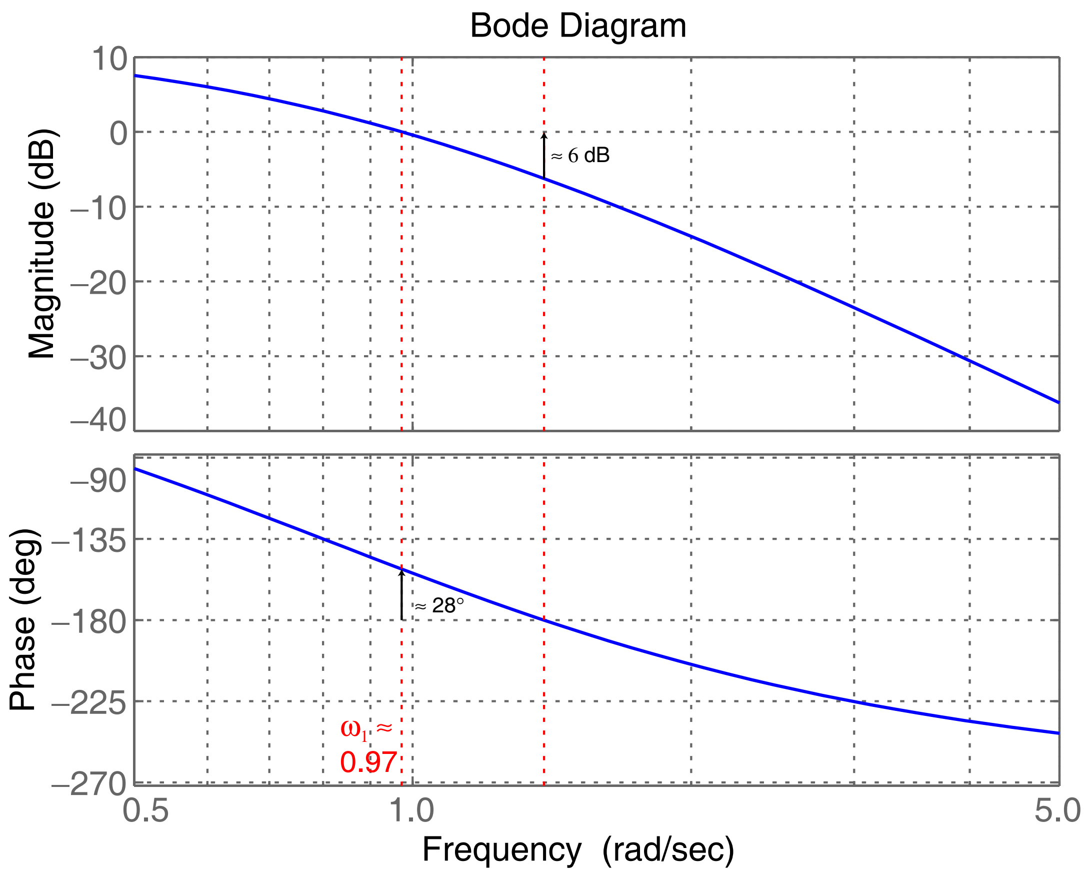
$$T_r = \frac{\tau_0\left[32\nu_0 + 6\tau_0\right]}{13\nu_0 + 8\tau_0} = 0.2003, \text{ and}$$

$$T_d = \frac{4\tau_0\nu_0}{11\nu_0 + 2\tau_0} = 0.0302.$$

4. **For the controller and plant with the open-loop Bode diagram shown below,:**

   a. **estimate both the *stability gain margin* and the *stability phase margin*, and**

   b. **design a lead compensator $C_{\text{lead}}(s)$ to increase the phase margin.** **(20 pts)**



a. The stability gain margin is the difference in gain from 0 dB when the phase is −180°, which is approximately 6 dB.

The stability phase margin is the difference in phase from −180° when the gain is 0 dB, which is approximately 28°.

b. A lead compensator that gives an approximately 45° phase lead at the frequency of the stability phase margin $\omega_1 \approx 0.97$ can be created by setting $\tau_1 = 1/\omega_1 = 1.0309$ and $\tau_2 = \tau_1/10 = 0.1031$, giving the transfer function:

$$C_{\text{lead}}(s) = \frac{1.0309s+1}{0.1031s+1}.$$