ELEC ENG 4CL4 – Control System Design

Solutions to Homework Assignment #4

1. The nominal model for a plant is given by:

$$G_o(s) = \frac{1}{(s+4)(-s+2)}.$$

Assume that this plant has to be controlled in a one-d.o.f. feedback loop such that the closed-loop poles are all at $s=-3$. Using the pole placement method, choose an appropriate minimum degree $A_{cl}(s)$ and synthesize a *biproper* controller $C(s)$ that has integration (i.e., one pole at $s=0$). (25 pts)

We first notice that a minimum degree biproper controller (with integration) requires an $A_{cl}(s)$ of degree $4\,(=2n)$. We thus choose $A_{cl}(s)=(s+3)^4 = s^4+12s^3+54s^2+108s+81$, and the controller is of the form:

$$C(s) = \frac{p_2 s^2 + p_1 s + p_0}{s(l_1 s + l_0)}.$$

The associated Diophantine equation is:

$$(s^2+2s-8)s(l_1 s + l_0) + (-1)(p_2 s^2 + p_1 s + p_0) = s^4+12s^3+54s^2+108s+81,$$

producing the pole-assignment matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -8 & 2 & -1 & 0 & 0 \\ 0 & -8 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_0 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 54 \\ 108 \\ 81 \end{bmatrix} \Rightarrow \begin{bmatrix} l_1 \\ l_0 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -42 \\ -188 \\ -81 \end{bmatrix}.$$

We thus obtain:

$$C(s) = \frac{-(42s^2+188s+81)}{s(s+10)}.$$

2. **Consider the nominal plant model:**

$$G_o(s) = \frac{1}{(s+1)^2}.$$

Using the pole placement method, design a *strictly proper* controller (i.e., it should have more poles than zeros) that gives the characteristic closed-loop polynomial $A_{cl}(s) = (s^2+4s+9)(s+2)^k$, where you should choose k to be the smallest integer that gives an $A_{cl}(s)$ of the appropriate degree. **(25 pts)**

To achieve a strictly proper controller the degree of $A_{cl}(s)$ has to be chosen larger than the minimum degree. In this case, the minimum degree is $3\,(=2n-1)$. Say we choose $k=2$, then the degree of $A_{cl}(s)$ turns out to be 4, and the controller is of the form:

$$C(s) = \frac{p_1 s + p_0}{l_2 s^2 + l_1 s + l_0}.$$

The Diophantine equation then becomes:

$$(s+1)^2\left(l_2 s^2 + l_1 s + l_0\right) + p_1 s + p_0 = \left(s^2+4s+9\right)\left(s^2+4s+4\right) = s^4 + 8s^3 + 29s^2 + 52s + 36,$$

producing the pole-assignment matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \\ l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 29 \\ 52 \\ 36 \end{bmatrix} \Rightarrow \begin{bmatrix} l_2 \\ l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 16 \\ 14 \\ 20 \end{bmatrix}.$$

We thus obtain:

$$C(s) = \frac{14s+20}{s^2+6s+16}.$$

3. The nominal model for a plant is given by:

$$G_o(s) = \frac{10(s-2)}{(s+2)(s-4)}.$$

a. **List the constraints that the poles and zeros of this plant model place on the appropriate bandwidth of a one-d.o.f., unity-feedback control loop.**

b. **Given these constraints, what would you consider an appropriate closed-loop bandwidth?** **(25 pts)**

a. The two constraints that the poles and zeros of this plant model place on the appropriate bandwidth of the closed-loop system are the following:

- The open-loop NMP zero at $s = z_o = 2$ sets an *upper bound* for the closed-loop bandwidth, since the integral constraint $\int_0^\infty y(t)\mathrm{e}^{-z_o t}\mathrm{d}t = \int_0^\infty y(t)\mathrm{e}^{-2t}\mathrm{d}t = 0$ says that a plant output which settles much faster than e^{-2t} will exhibit a *large undershoot*.

- The unstable open-loop pole at $s = \eta_o = 4$ sets a *lower bound* for the closed loop bandwidth, since the integral constraint $\int_0^\infty e(t)\mathrm{e}^{-\eta_o t}\mathrm{d}t = \int_0^\infty e(t)\mathrm{e}^{-4t}\mathrm{d}t = 0$ says that a plant output which settles much slower than e^{-4t} will exhibit a *large overshoot*.

b. This case is especially difficult because of the contradicting requirements of a bandwidth less than 2 rad/s to avoid large undershoot and a bandwidth greater than 4 rad/s to avoid large overshoot. Consequently, some compromise must be made. One possibility is to decide which of these two problems (i.e., undershoot or overshoot) is more detrimental to the plant's performance, taking into account such factors as safety, quality of the output product, and plant maintenance, and set the bandwidth to avoid the more detrimental problem and to permit the less detrimental problem. If both problems are of equal significance, then another possibility would be to set the close-loop bandwidth so that the undershoot and overshoot are of similar magnitudes. This is referred to as *design homogeneity*.

4. Consider a one-d.o.f., unity-feedback loop incorporating a controller *with integration* and the nominal plant model:

$$G_o(s) = \frac{4}{(s+2)^2}.$$

Show that the output of the control loop $y(t)$ must exhibit overshoot in response to the reference signal $r(t) = te^{-t/2}, \quad t \geq 0$. (25 pts)

We first note that the nominal plant model does not have any zeros, unstable poles or complex-conjugate poles that could contribute to overshoot. Therefore, the overshoot must be caused by the incorporation of integration in the controller. In a similar fashion to Lemma 8.1 of Goodwin et al., we assume that the open-loop plant and controller satisfy:

$$\begin{aligned}
A_o(s)L(s) &= s^i A_o(s)\bar{L}(s), \quad i \geq 1, \\
\lim_{s\to 0}\left(A_o(s)\bar{L}(s)\right) &= c_0 \neq 0, \\
\lim_{s\to 0}\left(B_o(s)P(s)\right) &= c_1 \neq 0,
\end{aligned}$$

that is, the controller has i poles at the origin. Then, for the reference signal $r(t) = te^{-t/2}, \quad t \geq 0$, which has the Laplace transform:

$$R(s) = \frac{1}{\left(s+\frac{1}{2}\right)^2},$$

the Laplace transform of the error signal $e(t)$ is:

$$\begin{aligned}
E(s) &= S_o(s)R(s) \\
&= \frac{s^i A_o(s)\bar{L}(s)}{s^i A_o(s)\bar{L}(s) + B_o(s)P(s)} \times \frac{1}{\left(s+\frac{1}{2}\right)^2}.
\end{aligned}$$

Evaluating the Laplace transform as $s \to 0$, we obtain the integral constraint:

$$\begin{aligned}
\int_0^\infty e(t)e^{-0t}dt &= \lim_{s\to 0} E(s) \\
&= \lim_{s\to 0} \frac{s^i A_o(s)\bar{L}(s)}{s^i A_o(s)\bar{L}(s) + B_o(s)P(s)} \times \frac{1}{\left(s+\frac{1}{2}\right)^2} \\
&= \frac{0}{0+c_1} \times \frac{1}{\left(0+\frac{1}{2}\right)^2} \\
\int_0^\infty e(t)dt &= 0.
\end{aligned}$$

For a positive-going reference signal, the error starts out positive, so in order for the integral constraint above to be satisfied, the error $e(t) = r(t) - y(t)$ must become negative at some time, i.e., the plant output $y(t)$ must overshoot the reference $r(t)$.