ELEC ENG 4CL4 – Control System Design

Solutions to Homework Assignment #5

1. Consider the following recursive equation describing the relationship between the input $u[k]$ and the output $y[k]$ in a discrete-time (sampled-data) system:

$$y[k] + 0.5y[k-1] = 1.5u[k-1].$$

a. **Determine the shift form (*z*-domain) transfer function $G_q(z)$ describing this system.**

b. **Using the result from part a, compute the response of the system to the unit Kronecker delta $\delta_K[k]$. (20 pts)**

a. Taking the *z*-transform of the difference equation gives:

$$Y(z) + 0.5z^{-1}Y(z) = 1.5z^{-1}U(z)$$

$$\Rightarrow G_q(z) = \frac{Y(z)}{U(z)} = \frac{1.5z^{-1}}{1+0.5z^{-1}} = \frac{1.5}{z+0.5}, \quad |z| > 0.5.$$

b. The impulse response is obtained by taking the inverse *z*-transform of the transfer function:

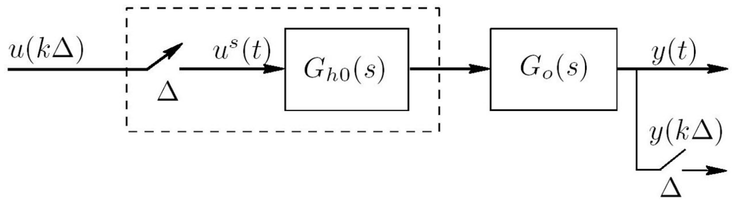
$$h[k] = \mathcal{Z}^{-1}\{G_q(z)\} = 1.5(-0.5)^{k-1}\mu[k-1], \quad k \ge 0,$$

$$\text{or} \quad -3(-0.5)^k \mu[k-1], \quad k \ge 0,$$

$$\text{or} \quad 3\delta_K[k] + 1.5(-0.5)^{k-1}, \quad k \ge 0,$$

$$\text{or} \quad 3\delta_K[k] - 3(-0.5)^k, \quad k \ge 0.$$

2. Assume that in the block diagram:



the continuous-time plant transfer function $G_o(s)$ is given by:

$$G_o(s) = \frac{1}{(s+1)^2}.$$

a. Compute the *discrete delta* transfer function from $u[k]$ to $y[k]$, $H_{o\delta}(\gamma)$, as a function of the sampling interval Δ.

b. Verify that, if we make $\Delta \to 0$, then:

$$\lim_{\Delta\to 0} H_{o\delta}(\gamma)\Big|_{\gamma=s} = G_o(s).$$ **(30 pts)**

a. One method for computing the delta form transfer function makes use of the property:

$$H_{o\delta}(\gamma) = H_{oq}(z)\Big|_{z=\Delta\gamma+1}.$$

The z-transform for the system's response to the Kronecker delta is:

$$\begin{aligned}
H_{oq}(z) &= \frac{z-1}{z}\mathcal{Z}\left[\mathfrak{L}^{-1}\left[\frac{1}{s(s+1)^2}\right]\Bigg|_{t=k\Delta}\right] \\
&= \frac{z-1}{z}\mathcal{Z}\left[1-\mathrm{e}^{-k\Delta}-k\Delta\,\mathrm{e}^{-k\Delta}\right] \\
&= \frac{z-1}{z}\left[\frac{z}{z-1}-\frac{z}{z-e^{-\Delta}}-\Delta\frac{e^{-\Delta}z}{\left(z-e^{-\Delta}\right)^2}\right] \\
&= \frac{z-1}{z}\left[\frac{z}{z-1}-\frac{z}{z-a}-\Delta\frac{az}{(z-a)^2}\right],
\end{aligned}$$

where $a = e^{-\Delta}$.

The delta form transfer function is then:

$$
\begin{aligned}
H_{o\delta}(\gamma) &= H_{oq}(z)\Big|_{z=\Delta\gamma+1} \\
&= \frac{\Delta\gamma}{\Delta\gamma+1}\left[\frac{\Delta\gamma+1}{\Delta\gamma} - \frac{\Delta\gamma+1}{\Delta\gamma+1-a} - \frac{a\Delta(\Delta\gamma+1)}{(\Delta\gamma+1-a)^2}\right] \\
&= 1 - \frac{\Delta\gamma}{\Delta\gamma+1-a} - \frac{a\Delta^2\gamma}{(\Delta\gamma+1-a)^2} \\
&= \frac{(1-a-a\Delta)\Delta\gamma + (1-a)^2}{(\Delta\gamma+1-a)^2} \\
&= \frac{\left(\frac{1-a}{\Delta} - a\right)\gamma + \left(\frac{1-a}{\Delta}\right)^2}{\left(\gamma + \frac{1-a}{\Delta}\right)^2} \\
&= \frac{(b-a)\gamma + b^2}{(\gamma+b)^2},
\end{aligned}
$$

where $b = \frac{1-e^{-\Delta}}{\Delta}$.

b. Taking the limit of a and b as $\Delta \to 0$ gives:

$$\lim_{\Delta\to 0} a = \lim_{\Delta\to 0} e^{-\Delta} = 1 \qquad \text{and} \qquad \lim_{\Delta\to 0} b = \lim_{\Delta\to 0} \frac{1-e^{-\Delta}}{\Delta} = 1,$$

and:

$$
\begin{aligned}
\lim_{\Delta\to 0} H_{o\delta}(\gamma)\Big|_{\gamma=s} &= \lim_{\Delta\to 0} \frac{(b-a)\gamma + b^2}{(\gamma+b)^2}\bigg|_{\gamma=s} \\
&= \frac{(1-1)\gamma + 1}{(\gamma+1)^2}\bigg|_{\gamma=s} \\
&= \frac{1}{(s+1)^2} \\
&= G_o(s).
\end{aligned}
$$

3. A continuous-time plant has the transfer function:

$$G_o(s) = \frac{1}{(s+1)^2}.$$

a. Synthesize a *minimal-prototype controller* $C_{1q}(z)$ <u>and</u> a *minimum-time dead-beat controller* $C_{2q}(z)$ for a sampling period Δ = 0.5 s.

b. For *each of these two controllers* placed in a one-d.o.f., unity-feedback control loop with the equivalent discrete-time (shift form) plant model $H_{oq}(z)$, evaluate the plant output $y[k]$ and the controller output $u[k]$ in response to a unit step reference. (50 pts)

a. From Q2 above, the equivalent *z*-transform transfer function, assuming a ZOH before $G_o(s)$, is:

$$H_{oq}(z) = \frac{z-1}{z}\left[\frac{z}{z-1} - \frac{z}{z-a} - \Delta\frac{az}{(z-a)^2}\right]$$

$$= 1 - \frac{z-1}{z-a} - \Delta\frac{a(z-1)}{(z-a)^2}$$

$$= \frac{(1-a-a\Delta)z + a^2 - a + a\Delta}{(z-a)^2}.$$

where $a = e^{-\Delta}$.

For $\Delta = 0.5$ s, $a = e^{-0.5} = 0.6065$ and:

$$H_{oq}(z) = \frac{0.0902z + 0.0646}{(z-0.6065)^2} = \frac{0.0902(z+0.7163)}{(z-0.6065)^2}.$$

The minimal-prototype controller $C_{1q}(z)$ is then:

$$C_{1q}(z) = \left[H_{oq}(z)\right]^{-1}\frac{1}{z-1} = \frac{(z-0.6065)^2}{0.0902(z+0.7163)}\frac{1}{z-1} = \frac{11.0860(z-0.6065)^2}{(z+0.7163)(z-1)}.$$

The minimum-time dead-beat controller $C_{2q}(z)$ is:

$$C_{2q}(z) = \frac{\alpha A_{oq}(z)}{z^2 - \alpha B_{oq}(z)} = \frac{\alpha(z-0.6065)^2}{z^2 - \alpha\cdot 0.0902(z+0.7163)},$$

where the value of α is:

$$\alpha = \frac{1}{B_{oq}(1)} = \frac{1}{0.0902+0.0646} = \frac{1}{0.1548} = 6.4592,$$

giving:

$$C_{2q}(z) = \frac{6.4592(z-0.6065)^2}{z^2 - 6.4592\cdot 0.0902(z+0.7163)} = \frac{6.4592(z-0.6065)^2}{z^2 - 0.5827z - 0.4173} = \frac{6.4592(z-0.6065)^2}{(z-1)(z+0.4173)}.$$

b. For the minimal prototype controller:

$$T_{oq}(z) = \frac{C_{1q}(z)H_{oq}(z)}{1+C_{1q}(z)H_{oq}(z)} = \frac{\frac{1}{z-1}}{1+\frac{1}{z-1}} = \frac{1}{z},$$

and:

$$S_{uoq}(z) = \frac{C_{1q}(z)}{1+C_{1q}(z)H_{oq}(z)} = \frac{\frac{11.0860(z-0.6065)^2}{(z+0.7163)(z-1)}}{1+\frac{1}{z-1}} = \frac{11.0860(z-0.6065)^2}{(z+0.7163)(z-1)} \cdot \frac{z-1}{z}.$$

The plant output is then:

$$Y_q(z) = T_{oq}(z)R(z) = \frac{1}{z}R(z)$$
$$\Rightarrow y[k] = \mu[k-1], \quad k \geq 0,$$

and the controller output is:

$$U_q(z) = S_{uoq}(z)R(z) = \frac{11.0860(z-0.6065)^2}{(z+0.7163)(z-1)} \cdot \frac{z-1}{\cancel{z}} \cdot \frac{\cancel{z}}{z-1} = 11.0860 + \frac{1}{z-1} - \frac{11.3024}{z+0.7163}$$
$$\Rightarrow u[k] = 11.0860\delta_K[k] + \left\{1 - 11.3024(-0.7163)^{k-1}\right\}\mu[k-1], \quad k \geq 0,$$
$$\text{or} \quad 11.0860\delta_K[k] + \mu[k-1] + 15.7789(-0.7163)^k \mu[k-1], \quad k \geq 0.$$

For the minimum-time dead-beat controller:

$$T_{oq}(z)=\frac{C_{2q}(z)H_{oq}(z)}{1+C_{2q}(z)H_{oq}(z)}$$

$$=\frac{\frac{6.4592\cancel{(z-0.6065)^2}}{(z-1)(z+0.4173)}\cdot\frac{0.0902(z+0.7163)}{\cancel{(z-0.6065)^2}}}{1+\frac{6.4592\cancel{(z-0.6065)^2}}{(z-1)(z+0.4173)}\cdot\frac{0.0902(z+0.7163)}{\cancel{(z-0.6065)^2}}}$$

$$=\frac{\frac{0.5826(z+0.7163)}{(z-1)(z+0.4173)}}{1+\frac{0.5826(z+0.7163)}{(z-1)(z+0.4173)}},$$

$$=\frac{0.5826(z+0.7163)}{\cancel{(z-1)(z+0.4173)}}\cdot\frac{\cancel{(z-1)(z+0.4173)}}{z^2}$$

$$=\frac{0.5826(z+0.7163)}{z^2}$$

and:

$$S_{uoq}(z)=\frac{C_{2q}(z)}{1+C_{2q}(z)H_{oq}(z)}=\frac{6.4592(z-0.6065)^2}{\cancel{(z-1)(z+0.4173)}}\cdot\frac{\cancel{(z-1)(z+0.4173)}}{z^2}$$

$$=\frac{6.4592(z-0.6065)^2}{z^2}.$$

The plant output is then:

$$Y_q(z)=T_{oq}(z)R(z)=\frac{0.5826(z+0.7163)}{z^{\cancel{2}}}\cdot\frac{\cancel{z}}{z-1}$$

$$=\frac{0.583(z+0.716)}{z(z-1)}=\frac{1}{z-1}-\frac{0.417}{z}\quad\text{or}\quad z^{-1}\frac{0.583z}{z-1}+z^{-2}\frac{0.417z}{z-1}$$

$$\Rightarrow y[k]=-0.417\delta_K[k-1]+\mu[k-1],\quad k\geq 0,$$

$$or\quad 0.583\mu[k-1]+0.417\mu[k-2],\quad k\geq 0,$$

and the controller output is:

$$U_q(z) = S_{uoq}(z)R(z) = \frac{6.4592(z-0.6065)^2}{z^{\not{2}}} \cdot \frac{\not{z}}{z-1}$$

$$= \frac{6.4592(z-0.6065)^2}{z(z-1)}$$

$$= 6.4592 + \frac{1}{z-1} - \frac{2.3760}{z} \quad \text{or} \quad \frac{6.4592z}{z-1} - z^{-1}\frac{7.8350z}{z-1} + z^{-2}\frac{2.3760z}{z-1}$$

$$\Rightarrow u[k] = 6.4592\delta_K[k] + \mu[k-1] - 2.3760\delta_K[k-1], \quad k \geq 0,$$

$$\text{or} \quad 6.4592\mu[k] - 7.8350\mu[k-1] + 2.3760\mu[k-2], \quad k \geq 0.$$