ELEC ENG 4CL4:

Control System Design

Notes for Lecture #11

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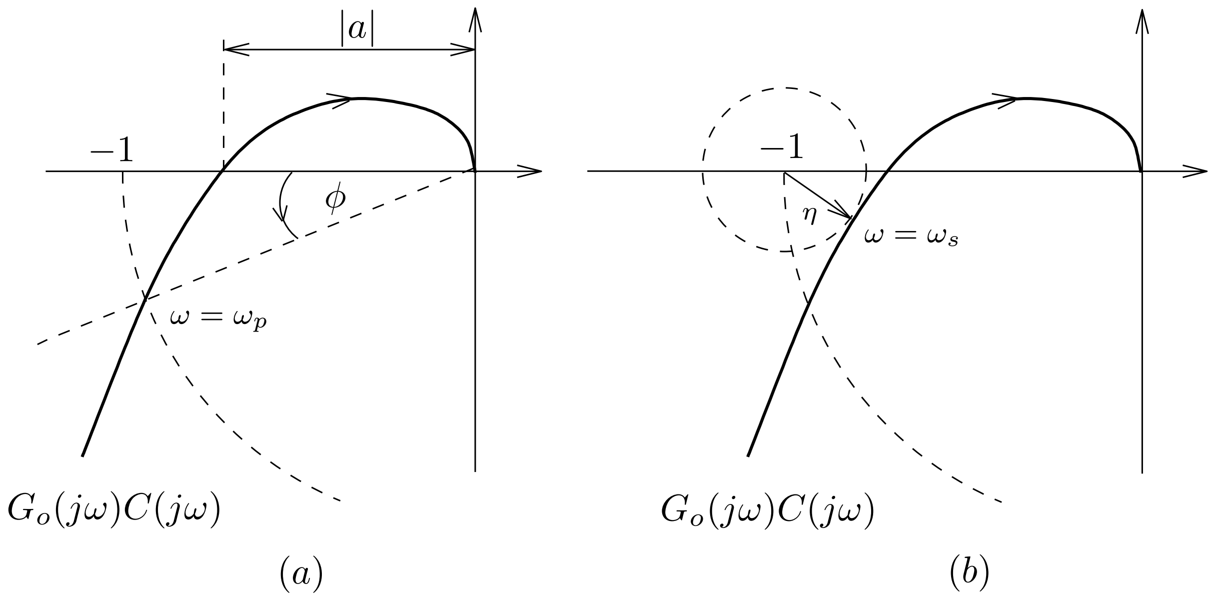
Dr. Ian C. Bruce
Room: CRL-229
Phone ext.: 26984
Email: ibruce@mail.ece.mcmaster.ca

Relative Stability: Stability margins and Sensitivity Peaks

In control system design, one often needs to go beyond the issue of closed loop stability. In particular, it is usually desirable to obtain some quantitative measures of how far from instability the nominal loop is, i.e. to quantify relative stability. This is achieved by introducing measures which describe the distance from the nominal open loop frequency response to the critical stability point (-1,0).

Figure 5.7: *Stability margins and sensitivity peak*



Gain and Phase Margins *Peak Sensitivity*

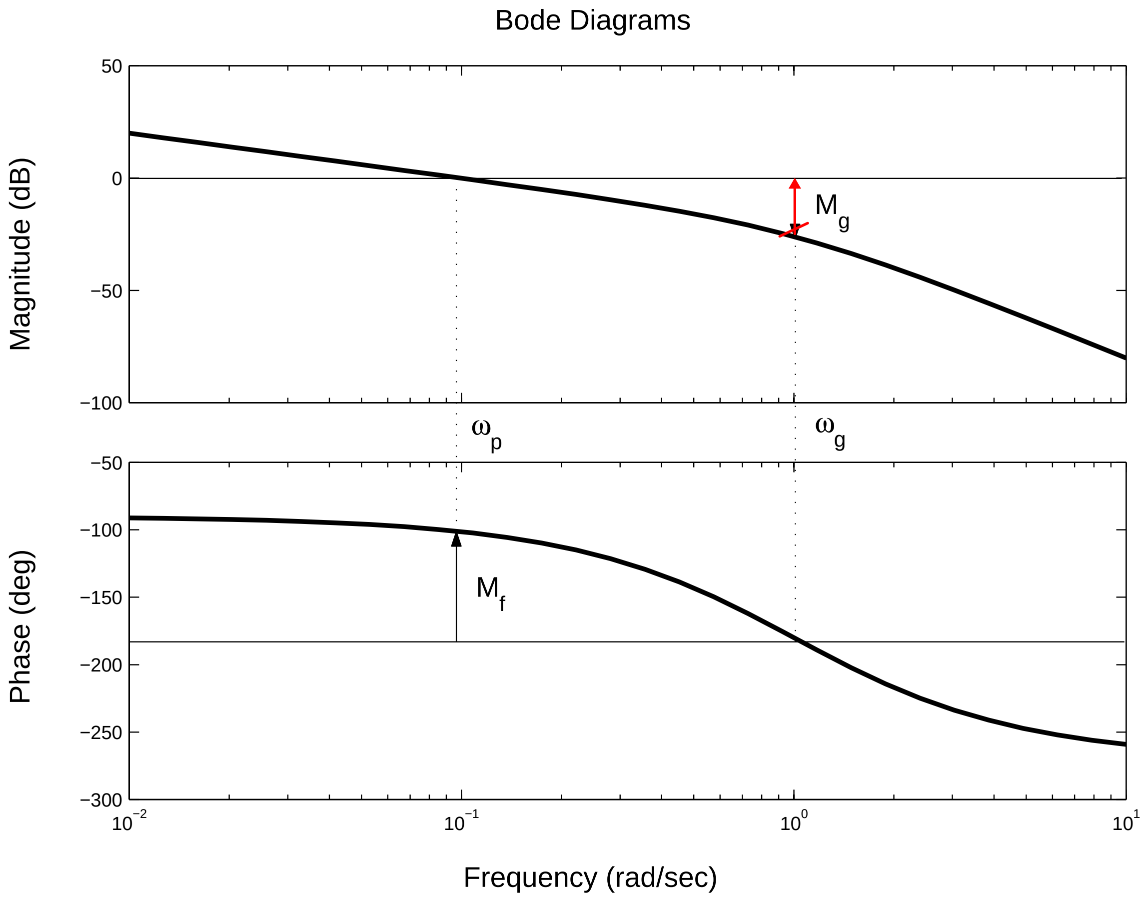
(a) The gain margin, M_g, and the phase margin M_f are defined as follows (see Figure 5.7):

$$M_g \triangleq -20\log_{10}(|a|)$$
$$M_f \triangleq \phi$$

(b) Peak sensitivity:

Since $S_0 = \dfrac{1}{1+G_0C}$, then S_0 is a maximum at the frequency where $G_0(jw)C(jw)$ is closest to the point -1. The peak sensitivity is thus $^1/_\eta$ - (see Figure 5.7).

Figure 5.8: *Stability margins in Bode diagrams*



Robustness

So far, we have only considered the effect that the controller has on the nominal closed loop formed with the nominal model for the plant. However, in practice, we are usually interested, not only in this nominal performance, but also the true performance achieved when the controller is applied to the true plant. This is the so called “Robustness” issue. We will show below that the nominal sensitivities do indeed tell us something about the true or achieved sensitivities.

Achieved Sensitivities

We contrast the nominal sensitivities derived previously with the achieved (or true) sensitivities when the controller $C(s)$ is applied to the calibration model, $G(s)$. This leads to the following calibration sensitivities:

$$
\begin{aligned}
T(s) &\triangleq \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{B(s)P(s)}{A(s)L(s)+B(s)P(s)} \\
S(s) &\triangleq \frac{1}{1+G(s)C(s)} = \frac{A(s)L(s)}{A(s)L(s)+B(s)P(s)} \\
S_i(s) &\triangleq \frac{G(s)}{1+G(s)C(s)} = \frac{B(s)L(s)}{A(s)L(s)+B(s)P(s)} \\
S_u(s) &\triangleq \frac{C(s)}{1+G(s)C(s)} = \frac{A(s)P(s)}{A(s)L(s)+B(s)P(s)}
\end{aligned}
$$

Relationship to Modelling Errors

The achieved sensitivity functions are given in terms of the nominal sensitivities as follows:

$$
\begin{aligned}
S(s) &= S_o(s)S_\Delta(s) \\
T(s) &= T_o(s)(1+G_\Delta(s))S_\Delta(s) \\
S_i(s) &= S_{io}(s)(1+G_\Delta(s))S_\Delta(s) \\
S_u(s) &= S_{uo}(s)S_\Delta(s) \\
S_\Delta(s) &= \frac{1}{1+T_o(s)G_\Delta(s)}
\end{aligned}
$$

Where $G_\Delta(s)$ is the multiplicative modelling error.

Robust Stability

We are concerned with the case where the nominal model and the true plant differ. It is then necessary that, in addition to nominal stability, we check that stability is retained when the true plant is controlled by the same controller. We call this property *robust stability*.

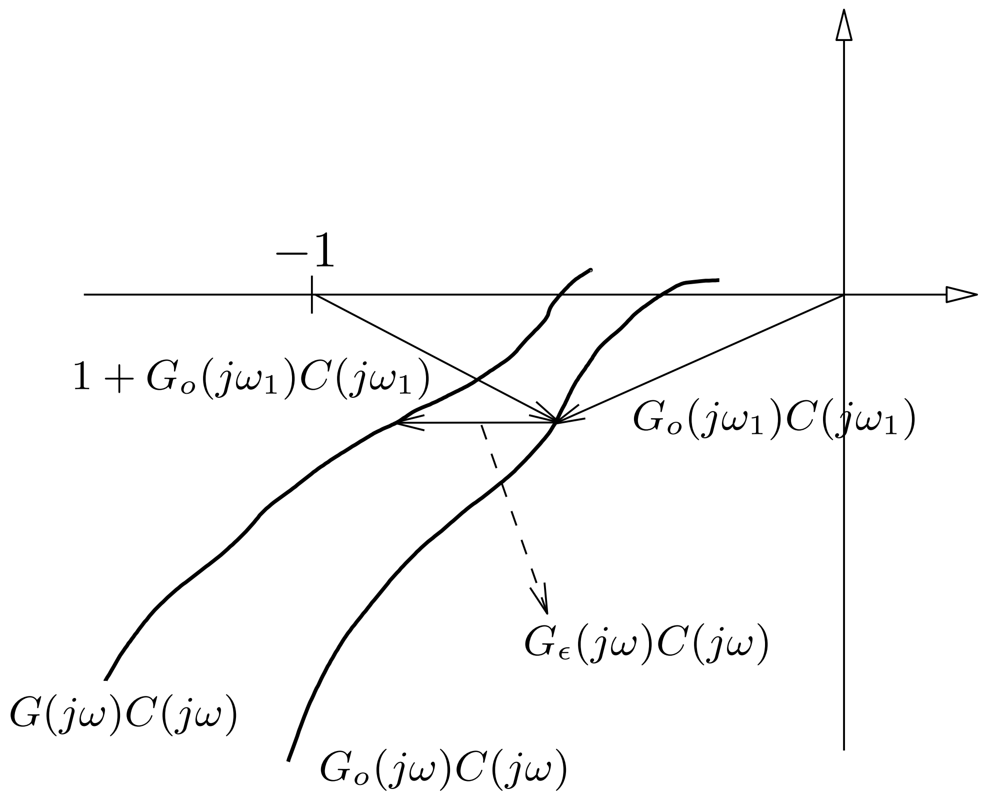
Theorem 5.3 (Robust stability theorem):

Consider a plant with nominal transfer function $G_0(s)$ and true transfer function given by $G(s)$. Assume that $C(s)$ is the transfer function of a controller which achieves nominal internal stability. Also assume that $G(s)C(s)$ and $G_0(s)C(s)$ have the same number of unstable poles. Then a sufficient condition for stability of the true feedback loop obtained by applying the controller to the true plant is that ...

$$|T_o(j\omega)||G_\Delta(j\omega)| = \left|\frac{G_o(j\omega)C(j\omega)}{1+G_o(j\omega)C(j\omega)}\right||G_\Delta(j\omega)| < 1 \qquad \forall\omega$$

where $G_\Delta(jw)$ is the frequency response of the multiplicative modeling error (MME).

Proof: *Consider the Nyquist plot for the nominal and the true loop*



From that figure we see that the same number of encirclements occur if

$$|G_\epsilon(j\omega)C(j\omega)| < |1 + G_o(j\omega)C(j\omega)| \qquad \forall\omega$$

this is equivalent to

$$\frac{|G_\Delta(j\omega)G_o(j\omega)C(j\omega)|}{|1 + G_o(j\omega)C(j\omega)|} < 1$$

∇∇∇

Example

In a feedback control loop, the open loop transfer function is given by

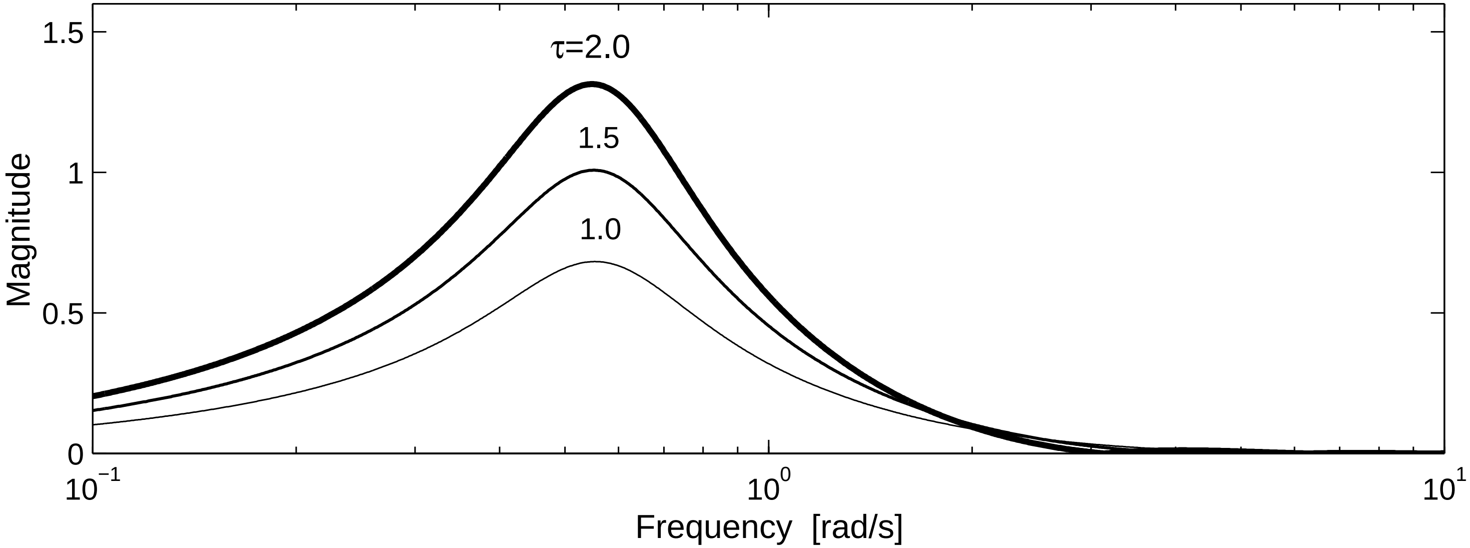
$$G_o(s)C(s) = \frac{0.5}{s(s+1)^2}$$

and the true plant transfer function is

$$G(s) = e^{-s\tau}G_o(s)$$

Use the Robust Stability Theorem to obtain a bound on the (unmodelled delay) τ which guarantees closed loop stability.

Figure 5.10: *Magnitude of the frequency response of $T_0(s)G_\Delta(s)$ for different values of τ*



Note that $|T_0(jw)G_\Delta(jw)| < 1, \ \forall w$ for $\tau \leq 1.5$.

Summary

- This chapter introduced the fundamentals of SISO feedback control loop analysis.

- Feedback introduces a cyclical dependence between controller and system:
  - the controller action affects the systems outputs,
  - and the system outputs affect the controller action.

- Well designed, feedback can
  - make an unstable system stable;
  - increase the response speed;
  - decrease the effects of disturbances;
  - decrease the effects of system parameter uncertainties, and more.

- Poorly designed, feedback can
  - introduce instabilities into a previously stable system;
  - add oscillations into a previously smooth response;
  - result in high sensitivity to measurement noise;
  - result in sensitivity to structural modeling errors, and more.