ELEC ENG 4CL4:

Control System Design

Notes for Lecture #17

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Chapter 7

Synthesis of SISO Controllers

Pole Assignment

In the previous chapter, we examined PID control. However, the tuning methods we used were essentially ad-hoc. Here we begin to look at more formal methods for control system design. In particular, we examine the following key synthesis question:

> *Given a model, can one systematically synthesize a controller such that the closed loop poles are in predefined locations*?

This chapter will show that this is indeed possible. We call this *pole assignment*, which is a fundamental idea in control synthesis.

Polynomial Approach

In the nominal control loop, let the controller and nominal model transfer functions be respectively given by:

$$C(s) = \frac{P(s)}{L(s)} \qquad G_o(s) = \frac{B_o(s)}{A_o(s)}$$

with

$$P(s) = p_{n_p} s^{n_p} + p_{n_p-1} s^{n_p-1} + \ldots + p_0$$
$$L(s) = l_{n_l} s^{n_l} + l_{n_l-1} s^{n_l-1} + \ldots + l_0$$
$$B_o(s) = b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \ldots + b_0$$
$$A_o(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

Consider now a desired closed loop polynomial given by

$$A_{cl}(s) = a^c_{n_c} s^{n_c} + a^c_{n_c - 1} s^{n_c - 1} + \ldots + a^c_0$$

Goal

Our objective here will be to see if, for given values of B_0 and A_0, P and L can be designed so that the closed loop characteristic polynomial is $A_{cl}(s)$.

We will see that, under quite general conditions, this is indeed possible.

Before delving into the general theory, we first examine a simple problem to illustrate the ideas.

Example 7.1

Let $G_0(s) = B_0(s)/A_0(s)$ be the nominal model of a plant with $A_0(s) = s^2 + 3s + 2$, $B_0(s) = 1$ and consider a controller of the form:

$$C(s) = \frac{P(s)}{L(s)}; \qquad P(s) = p_1 s + p_0; \qquad L(s) = l_1 s + l_0$$

We see that the closed loop characteristic polynomial satisfies:

$$A_0(s)L(s) + B_0(s)P(s) = (s^2 + 3s + 2)\,(l_1 s + l_0) + (p_1 s + p_0)$$

Say that we would like this to be equal to a polynomial $s^3 + 3s^2 + 3s + 1$, then equating coefficients gives:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

It is readily verified that the 4×4 matrix above is nonsingular, meaning that we can solve for l_1, l_0, p_1 and p_0 leading to $l_1 = 1$, $l_0 = 0$, $p_1 = 1$ and $p_0 = 1$. Hence the desired characteristic polynomial is achieved using the controller $C(s) = (s + 1)/s$.

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We next turn to the general case. We first note the following mathematical result.

Sylvester's Theorem

Consider two polynomials

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$
$$B(s) = b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0$$

Together with the following eliminant matrix:

$$\mathbf{M_e} = \begin{bmatrix} a_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_n & b_0 & b_1 & \cdots & b_n \\ 0 & a_0 & \cdots & a_{n-1} & 0 & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{bmatrix}$$

Then $A(s)$ and $B(s)$ are relatively prime (coprime) if and only if $\det(\mathbf{M_e}) \neq 0$.

Application of Sylvester's Theorem

We will next use the above theorem to show how closed loop pole-assignment is possible for general linear single-input single-output systems.

In particular, we have the following result:

Lemma 7.1: (SISO pole placement. Polynomial approach). Consider a one d.o.f. feedback loop with controller and plant nominal model given by (7.2.2) to (7.2.6). Assume that $B_0(s)$ and $A_0(s)$ are relatively prime (coprime), i.e. they have no common factors. Let $A_{cl}(s)$ be an arbitrary polynomial of degree $n_c = 2n - 1$. Then there exist polynomials $P(s)$ and $L(s)$, with degrees $n_p = n_l = n - 1$ such that

$$A_o(s)L(s) + B_o(s)P(s) = A_{cl}(s)$$

The above result shows that, in very general situations, pole assignment can be achieved.

We next study some special cases where additional constraints are placed on the solutions obtained.

Constraining the Solution

Forcing integration in the loop: A standard requirement in control system design is that, in steady state, the nominal control loop should yield zero tracking error due to D.C. components in either the reference, input disturbance or output disturbance. For this to be achieved, a necessary and sufficient condition is that the nominal loop be internally stable and that the controller have, at least, one pole at the origin. This will render the appropriate sensitivity functions zero at zero frequency.

To achieve this we choose

$$L(s) = s\bar{L}(s)$$

The closed loop equation can then be rewritten as

$$\bar{A}_o(s)\bar{L}(s) + B_o(s)P(s) = A_{cl}(s) \qquad \text{with} \qquad \bar{A}_o(s) \stackrel{\triangle}{=} sA_o(s)$$