ELEC ENG 4CL4:

Control System Design

Notes for Lecture #26

Monday, March 15, 2004

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The current chapter is principally concerned with modelling issues, i.e. how to relate samples of the output of a physical system to the sampled data input.

Specific topics to be covered are:

- Discrete-time signals
- Z-transforms and Delta transforms
- Sampling and reconstruction
- Aliasing and anti-aliasing filters
- Sampled-data control systems

Sampling

The result of sampling a continuous time signal is shown below:

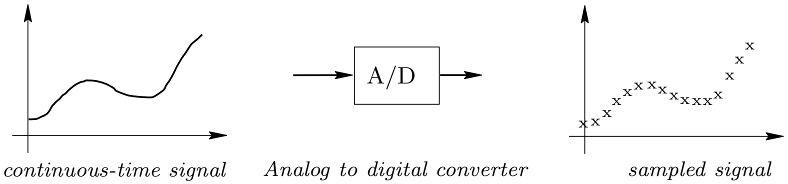


Figure 12.10: *The result of sampling*

There will always be loss of information due to sampling. However, the extent of this loss depends on the sampling method and the associated parameters. For example, assume that a sequence of samples is taken of a signal $f(t)$ every Δ seconds, then the sampling frequency needs to be large enough in comparison with the maximum rate of change of $f(t)$. Otherwise, high frequency components will be mistakenly interpreted as low frequencies in the samples sequence.

Example 12.1

Consider the signal

$$f(t) = 3\cos 2\pi t + \cos\left(20\pi t + \frac{\pi}{3}\right)$$

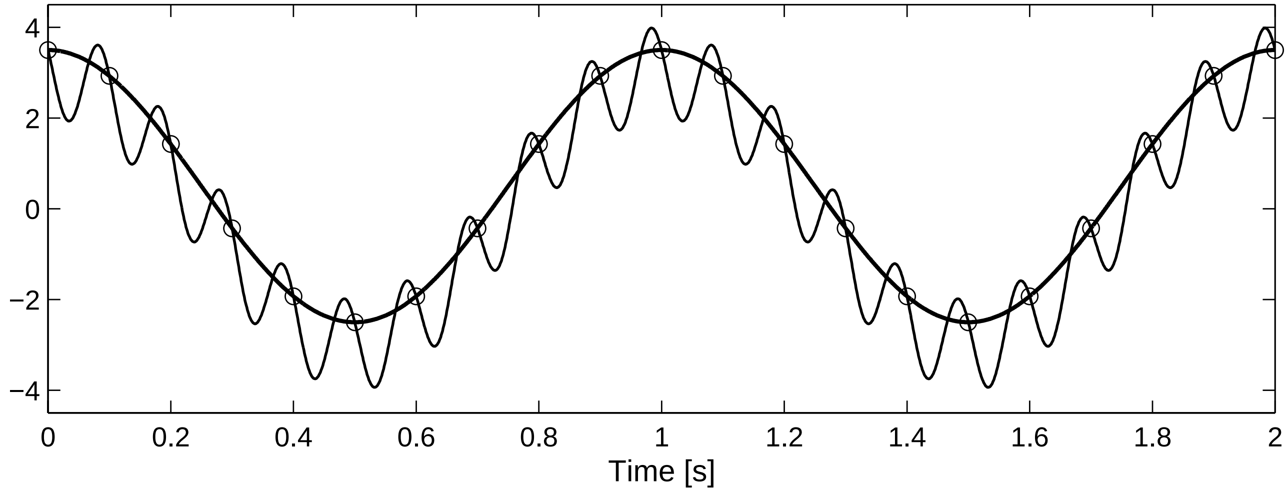
We observe that if the sampling period Δ is chosen equal to 0.1[*s*] then

$$\begin{aligned} f(k\Delta) &= 3\cos(0.2k\pi) + \cos\left(2k\pi + \frac{\pi}{3}\right) \\ &= 3\cos(0.2k\pi) + 0.5 \end{aligned}$$

from where it is evident that the high frequency component has been shifted to a constant, i.e. the high frequency component appears as a signal of low frequency (*here zero*). This phenomenon is known as aliasing.

This effect is illustrated on the next slide.

Figure 12.1: *Aliasing effect when using low sampling rate*



Conclusion:

To mitigate the effect of aliasing the sampling rate must be high relative to the rate of change of the signals of interest. A typical rule of thumb is to require that the sampling rate be 5 to 10 times the bandwidth of the signals.

Signal Reconstruction

The output of a digital controller is another sequence of numbers $\{u[k]\}$ which are the sample values of the intended control signal. These sample values need to be converted back to continuous time functions before they can be applied to the plant. Usually, this is done by interpolating them into a staircase function $u(t)$ as illustrated in Figure 12.2.

Figure 12.2: *Staircase reconstruction*

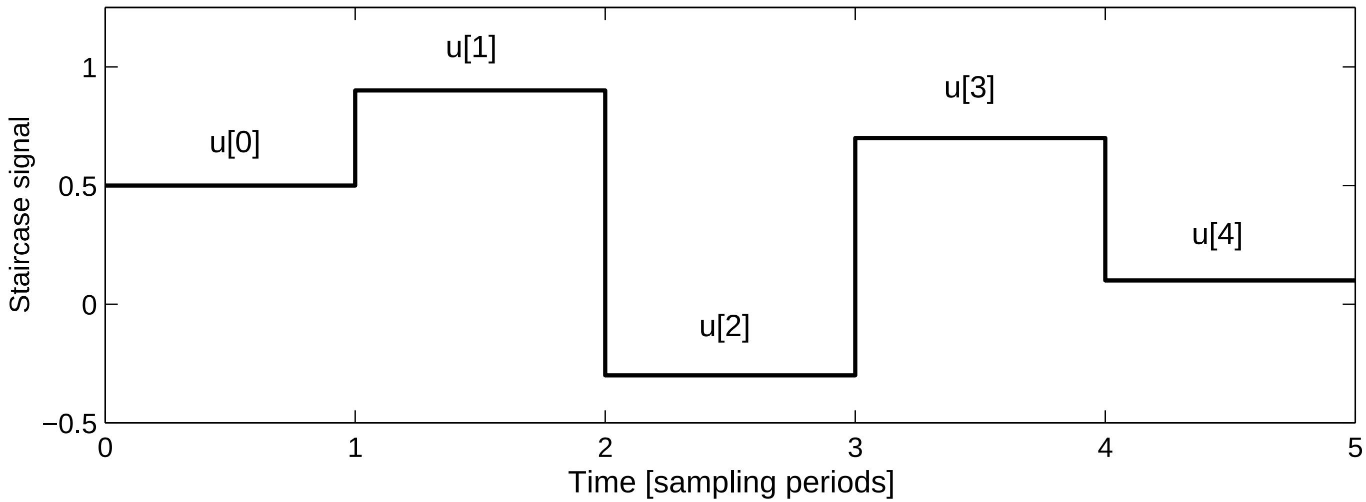


Illustration of Signal Reconstruction

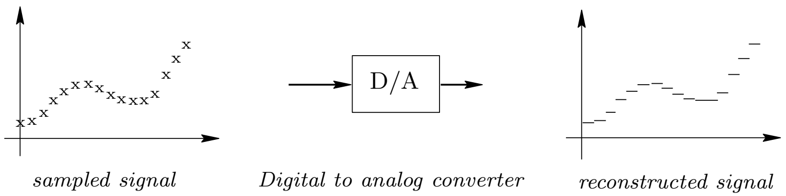


Figure 12.11: *The result of reconstruction*

Modelling

Given the process of signal reconstruction and sampling, we see that the net result is that, inside the computer, the system input and output simply appear as sequences of numbers.

It therefore makes sense to build digital models that relate a discrete time input sequence, $\{u(k)\}$, to a sampled output sequence $\{y(k\Delta)\}$.

Linear Discrete Time Models

A useful discrete time model of the type referred to above is the linear version of the high order difference equation model. In the discrete case, this model takes the form:

$$y[k+n] + \overline{a}_{n-1}y[k+n-1] + \cdots + \overline{a}_0 y[k]$$
$$= \overline{b}_{n-1}u[k+n-1] + \cdots + \overline{b}_0 u[k]$$

Note that we saw a special form of this model in relation to the motivational servo example presented earlier.

To simplify the way we write the model equations, we will find it useful to have a simple notation to represent a time-shifted output sample, $y\left(\overline{k+m}\Delta\right)$. We introduce a special operator (the shift operator) that allows us to write this very compactly.

The Shift Operator

Forward shift operator

$$\boxed{q(f[k]) \triangleq f[k+1]}$$

In terms of this operator, the model given earlier becomes:

$$q^n y[k] + \overline{a}_{n-1} q^{n-1} y[k] + \cdots + \overline{a}_0 y[k] = \overline{b}_m q^m u[k] + \cdots + \overline{b}_0 u[k]$$

For a discrete time system it is also possible to have discrete state space models. In the shift domain these models take the form:

$$qx[k] = \mathbf{A}_q x[k] + \mathbf{B}_q u[k]$$
$$y[k] = \mathbf{C}_q x[k] + \mathbf{D}_q u[k]$$

Z-Transform

Analogously to the use of Laplace Transforms for continuous time signals, we introduce the Z-transform for discrete time signals.

Consider a sequence $\{y[k]; k = 0, 1, 2, \ldots]$. Then the Z-transform pair associated with $\{y[k]\}$ is given by

$$\mathcal{Z}\left[y[k]\right] = Y(z) = \sum_{k=0}^{\infty} z^{-k} y[k]$$

$$\mathcal{Z}^{-1}\left[Y(z)\right] = y[k] = \frac{1}{2\pi j} \oint z^{k-1} Y(z) dz$$

A table of Z-transforms of typical sequences is given in Table 12.1 (*see the next slide*).

Also, a table of Z-transform properties is given in Table 12.2 (*see the slide after next*).

$f[k]$	$\mathcal{Z}[f[k]]$	Region of convergence
1	$\frac{z}{z-1}$	$\|z\|>1$
$\delta_K[k]$	1	$\|z\|>0$
k	$\frac{z}{(z-1)^2}$	$\|z\|>1$
k^2	$\frac{z(z+1)}{(z-1)^3}$	$\|z\|>1$
a^k	$\frac{z}{z-a}$	$\|z\|>\|a\|$
ka^k	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$\cos k\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$	$\|z\|>1$
$\sin k\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$	$\|z\|>1$
$a^k\cos k\theta$	$\frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$	$\|z\|>a$
$a^k\sin k\theta$	$\frac{az\sin\theta)}{z^2-2az\cos\theta+a^2}$	$\|z\|>a$
$k\cos k\theta$	$\frac{z(z^2\cos\theta-2z+\cos\theta)}{z^2-2z\cos\theta+1}$	$\|z\|>1$
$\mu[k]-\mu[k-k_o],\quad k_o\in\mathbb{N}$	$\frac{1+z+z^2+\ldots+z^{k_o-1}}{z^{k_o-1}}$	$\|z\|>0$

Table 12.1: *Z-transform table*

$f[k]$	$\mathcal{Z}[f[k]]$	Names
$\sum_{i=1}^{l} a_i f_i[k]$	$\sum_{i=1}^{l} a_i F_i(z)$	Partial fractions
$f[k+1]$	$zF(z) - zf(0)$	Forward shift
$\sum_{l=0}^{k} f[l]$	$\frac{z}{z-1}F(z)$	Summation
$f[k-1]$	$z^{-1}F(z) + f(-1)$	Backward shift
$y[k-l]\mu[k-l]$	$z^{-l}Y(z)$	Unit step
$kf[k]$	$-z\frac{dF(z)}{dz}$	
$\frac{1}{k}f[k]$	$\int_z^{\infty} \frac{F(\zeta)}{\zeta} d\zeta$	
$\lim_{k\to\infty} y[k]$	$\lim_{z\to 1}(z-1)Y(z)$	Final value theorem
$\lim_{k\to 0} y[k]$	$\lim_{z\to\infty} Y(z)$	Initial value theorem
$\sum_{l=0}^{k} f_1[l]f_2[k-l]$	$F_1(z)F_2(z)$	Convolution
$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1(\zeta)F_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}$	Complex convolution
$(\lambda)^k f_1[k]$	$F_1\left(\frac{z}{\lambda}\right)$	Frequency scaling

Table 12.2: *Z-transform properties. Note that $F_i(z) = Z[f_i[k]]$, $\mu[k]$ denotes, as usual, a unit step, $y[\infty]$ must be well defined and the convolution property holds provided that $f_1[k] = f_2[k] = 0$ for all $k < 0$.*

How do we use Z-transforms ?

We saw earlier that Laplace Transforms have a remarkable property that they convert differential equations into algebraic equations.

Z-transforms have a similar property for discrete time models, namely they convert difference equations (expressed in terms of the shift operator q) into algebraic equations.

We illustrate this below for a discrete high-order difference equation model:

Discrete Transfer Functions

Taking Z-transforms on each side of the high order difference equation model leads to

$$A_q(z)Y_q(z) = B_q(z)U_q(z) + f_q(z, x_o)$$

where $Y_q(z)$, $U_q(z)$ are the Z-transform of the sequences $\{y[k]\}$ and $\{u[k]\}$ respectively, and

$$A_q(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_o$$
$$B_q(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_o$$

We then see that (*ignoring the initial conditions*) the Z-transform of the output $Y(z)$ is related to the Z-transform of the input by $Y(z) = G_q(z)U(z)$ where

$$G_q(z) \triangleq \frac{B_q(z)}{A_q(z)}$$

$G_q(z)$ is called the *discrete* (*shift form*) transfer function.

An interesting observation

We see from Table 12.1 that the Z-transform of a unit pulse is 1. Also, we have just seen that Z-transform of the output of discrete linear systems satisfies

$$Y(z) = G_q(z)\mathrm{U}(z)$$

where $G_q(z)$ is the transfer function and $U(z)$ the input.

Hence, the transfer function is the Z-transform of the output when the input is a Kronecker delta.

Example:

Find the unit step response of a system with transfer function given by

$$G_q(z) = \frac{0.5}{z + 0.8} \quad 0.5$$

Solution: The Z-transform of the step response, *y*[*k*], is given by

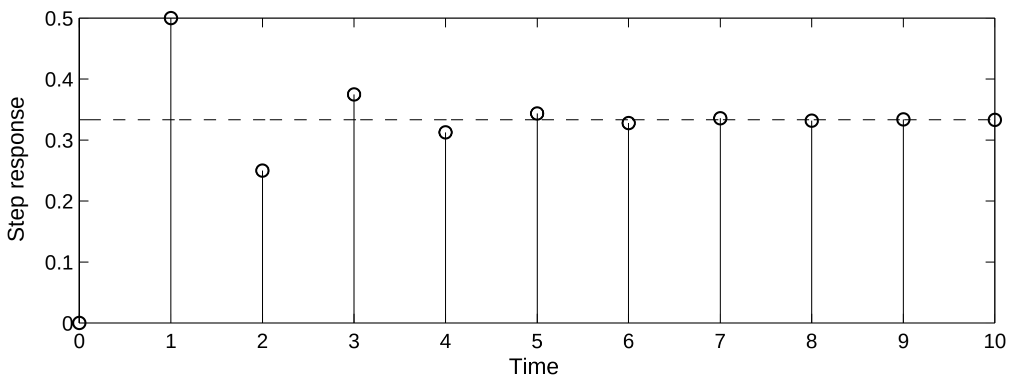
$$Y_q(z) = \frac{0.5}{z + 0.5} U_q(z) = \frac{0.5z}{(z + 0.5)(z - 1)}$$

Expanding in partial fractions (use MATLAB command **residue**) we obtain

$$Y_q(z) = \frac{z}{3(z-1)} - \frac{z}{3(z+0.5)} \Longleftrightarrow y[k] = \frac{1}{3}\left(1 - (-0.5)^k\right)\mu[k]$$

The response is shown on the next slide.

Figure 12.3: *Unit step response of a system exhibiting ringing response*



Note that the response contains the term $(-0.5)^k$, which corresponds to an oscillatory behavior (known as ringing). In discrete time this can occur (as in this example) for a single negative real pole whereas, in continuous time, a pair of complex conjugate poles are necessary to produce this effect.