ELEC ENG 4CL4:

Control System Design

Notes for Lecture #28

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Using Continuous State Space Models

Next we show how a discrete model can be developed when the plant is described by a continuous time state space model

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t)$$

Then, using the solution formula (*see Chapter 3*) the sampled state response over an interval Δ is given by

$$x((k+1)\Delta) = e^{\mathbf{A}\Delta}x(k\Delta) + \int_0^{\Delta} e^{\mathbf{A}(\Delta-\tau)}\mathbf{B}u(\tau)d\tau$$

Now using the fact that $u(\tau+k\Delta)$ is equal to $u(k\Delta)$ for $0 \leq \tau < \Delta$ we have

$$x((k+1)\Delta) = \mathbf{A}_q x(k\Delta) + \mathbf{B}_q u(k\Delta)$$

where

$$\mathbf{A}_q = e^{\mathbf{A}\Delta}$$
$$\mathbf{B}_q = \int_0^{\Delta} e^{\mathbf{A}(\Delta-\tau)}\mathbf{B}d\tau$$

Also the output is

$$y(k\Delta) = \mathbf{C}_q x(k\Delta) \qquad \text{where} \quad \mathbf{C}_q = \mathbf{C}$$

Shift form

The discrete time state space model derived above can be expressed compactly using the forward shift operator, q, as

$$
\begin{aligned}
qx[k] &= \mathbf{A}_q x[k] + \mathbf{B}_q u[k] \\
y[k] &= \mathbf{C}_q x[k]
\end{aligned}
$$

where

$$\mathbf{A}_q \stackrel{\triangle}{=} e^{\mathbf{A}\Delta} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}\Delta)^k}{k!}$$

$$\mathbf{B}_q \stackrel{\triangle}{=} \int_0^{\Delta} e^{\mathbf{A}(\Delta-\tau)}\mathbf{B}d\tau = \mathbf{A}^{-1}\left[e^{\mathbf{A}\Delta} - I\right] \qquad \text{if} \quad \mathbf{A} \quad \text{is nonsingular}$$

$$\mathbf{C}_q \stackrel{\triangle}{=} \mathbf{C}$$

$$\mathbf{D}_q \stackrel{\triangle}{=} \mathbf{D}$$

Delta Form

Alternatively, the discrete state space model can be expressed in Delta form as

$$\begin{aligned}\delta x(t_k) &= \mathbf{A}_\delta x(t_k) + \mathbf{B}_\delta u(t_k)\\ y(t_k) &= \mathbf{C}_\delta x(t_k) + \mathbf{D}_\delta u(t_k)\end{aligned}$$

where $\mathbf{C}_\delta = \mathbf{C}_q = \mathbf{C}$, $\mathbf{D}_\delta = \mathbf{D}_q = \mathbf{D}$ and

$$\begin{aligned}\mathbf{A}_\delta &\triangleq \frac{e^{\mathbf{A}\Delta} - I}{\Delta}\\ \mathbf{B}_\delta &\triangleq \mathbf{\Omega}\mathbf{B}\\ \mathbf{\Omega} &= \frac{1}{\Delta}\int_0^\Delta e^{\mathbf{A}\tau}d\tau = I + \frac{\mathbf{A}\Delta}{2!} + \frac{\mathbf{A}^2\Delta^2}{3!} + \ldots\end{aligned}$$

Some Comparisons of Shift and Delta Forms

For the delta form we have

$$\lim_{\Delta \to 0} \mathbf{A}_\delta = \mathbf{A} \qquad \lim_{\Delta \to 0} \mathbf{B}_\delta = \mathbf{B}$$

For the shift form

$$\lim_{\Delta \to 0} \mathbf{A}_q = I \qquad \lim_{\Delta \to 0} \mathbf{B}_q = \mathbf{0}$$

Indeed, this reconfirms one of the principal advantages of the delta form, namely that it converges to the underlying continuous time model as the sampling period approaches zero. Note that this is not true of the alternative shift operator form.

Frequency Response of Sampled Data Systems

We next evaluate the frequency response of a linear discrete time system having transfer function $H_q(z)$. Consider a sine wave input given by

$$u(k\Delta) = \sin(\omega k\Delta) = \sin\left(2\pi k\frac{\omega}{\omega_s}\right) = \frac{1}{2j}\left(e^{j2\pi k\frac{\omega}{\omega_s}} - e^{-j2\pi k\frac{\omega}{\omega_s}}\right)$$

where $\omega_s = \frac{2\pi}{\Delta}$.

Following the same procedure as in the continuous time case (see Section 4.9) we see that the system output response to the input is

$$y(k\Delta) = \alpha(\omega)\sin(\omega k\Delta + \phi(\omega))$$

where

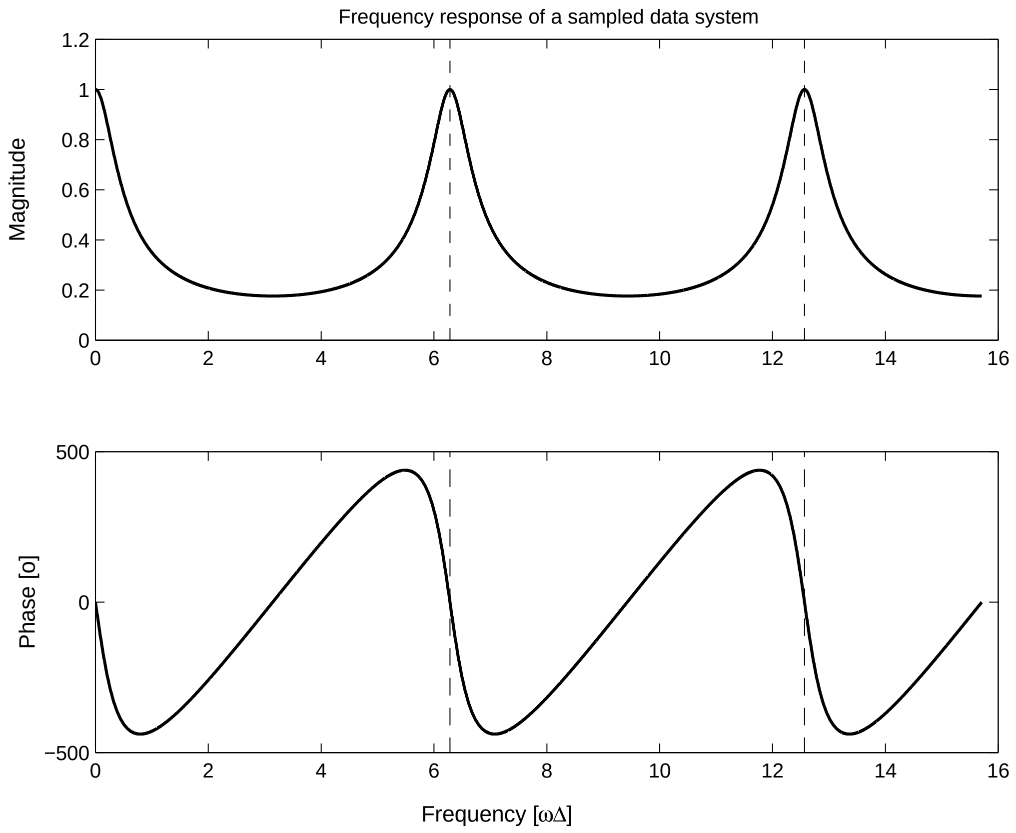
$$H_q(e^{j\omega\Delta}) = \alpha(\omega)e^{j\phi(\omega)}$$

The frequency response of a discrete time system depends upon $e^{j\omega\Delta}$ and is thus periodic in ω with period $^{2\pi}/_{\Delta}$.

The next slide illustrates this fact by showing the frequency response of

$$H_q[z] = \frac{0.3}{z - 0.7}$$

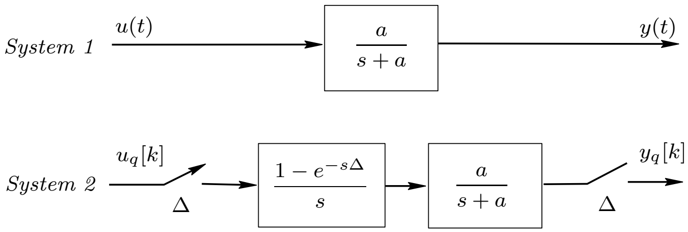
Figure 12.7: *Periodicity in the frequency response of sampled data systems.*



Another feature of particular interest is that the sampled data frequency response converges to its continuous counterpart as $\Delta \rightarrow 0$ and hence much insight can be obtained by simply looking at the continuous version. This is exemplified below.

Example 12.11: Consider the two systems shown in Figure 12.8 on the next page: Compare the frequency response of both systems in the range $[0, \omega_s]$.

Figure 12.8: *Continuous and sampled data systems*



The continuous time transfer function

$$H(s) = \frac{a}{s+a}$$

The continuous and discrete frequency responses are:

$$H(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{a}{j\omega + a}$$

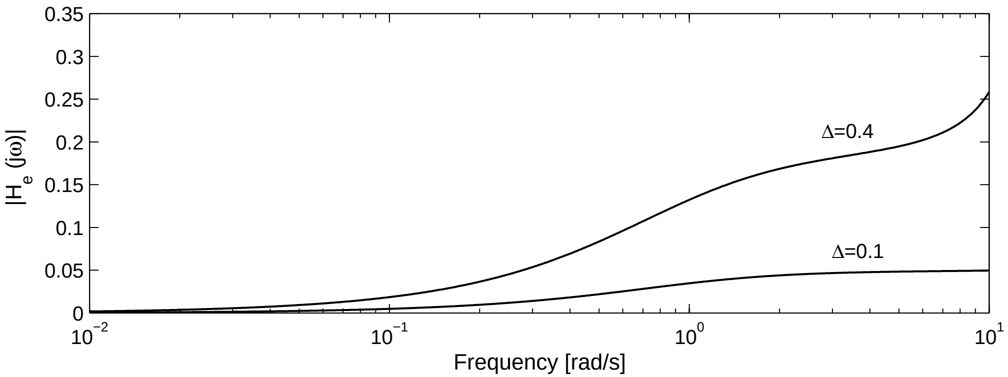
$$H_q\left(e^{j\omega\Delta}\right) = \frac{Y_q\left(e^{j\omega\Delta}\right)}{U_q\left(e^{j\omega\Delta}\right)} = Z\left\{G_{h0}(s)\frac{a}{s+a}\right\}\Big|_{z=e^{j\omega\Delta}} = \frac{1-e^{-a\Delta}}{e^{j\omega\Delta} - e^{-a\Delta}}$$

Note that for $\omega << \omega_s$ and $a << \omega_s$ *i.e.* $\omega\Delta << 1$ and $a\Delta << 1$, then we can use a first order Taylor's series approximation for the exponentials $e^{-a\Delta}$ and $e^{j\omega\Delta}$ in the discrete case leading to

$$H_q(j\omega\Delta) \approx \frac{1 - 1 + a\Delta}{1 + j\omega\Delta - 1 + a\Delta} = \frac{a}{j\omega + a} = H(j\omega)$$

The next slide compares the two frequency responses as a function of input frequency for two different values of Δ. Note that for Δ small, the two frequency responses are very close.

Figure 12.9: *Asymptotic behavior of a sampled data transfer function*



Summary

- Very few plants encountered by the control engineer are digital, most are continuous. That is, the control signal applied to the process, as well as the measurements received from the process, are usually continuous time.
- Modern control systems, however, are almost exclusively implemented on digital computers.
- Compared to the historical analog controller implementation, the digital computer provides
  - much greater ease of implementing complex algorithms,
  - convenient (graphical) man-machine interfaces,
  - logging, trending and diagnostics of internal controller and
  - flexibility to implement filtering and other forms of signal processing operations.

- Digital computers operate with sequences in time, rather than continuous functions in time.
  Therefore,
  - input signals to the digital controller-notably process measurements - must be sampled;
  - outputs from the digital controller-notably control signals - must be interpolated from a digital sequence of values to a continuous function in time.
- Sampling (see next slide) is carried out by A/D (analog to digital converters.

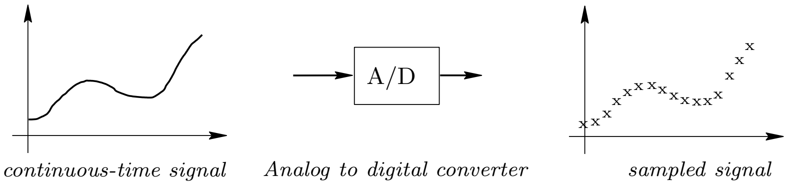


Figure 12.10: *The result of sampling*

- The converse, reconstructing a continuous time signal from digital samples, is carried out by D/A (digital to analog) converters. There are different ways of interpolating between the discrete samples, but the so called zero-order hold (see next slide) is by far the most common.

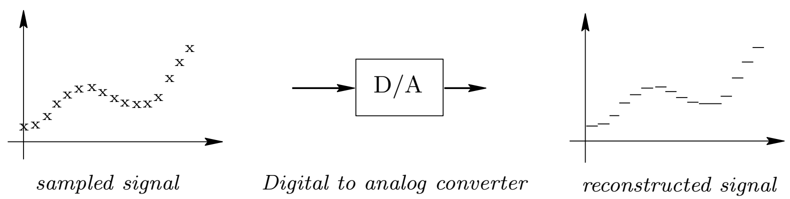


Figure 12.11: *The result of reconstruction*

- When sampling a continuous time signal,
  - an appropriate sampling rate must be chosen
  - an anti-aliasing filter (low-pass) should be included to avoid frequency folding.
- Analysis of digital systems relies on discrete time versions of the continuous operators.

- The chapter has introduced two discrete operators:
  - the shift operator, q, defined by $qx[k] \triangleq x[k+1]$
  - the δ-operator, δ, defined by $\delta x[k] \triangleq \frac{x[k+1]-x[k]}{\Delta}$
- Thus, $\delta = \frac{q-1}{\Delta}$, or $q = \delta\Delta + 1$.
- Due to this conversion possibility, the choice is largely based on preference and experience. Comparisons are outlined below.

- The shift operator, q,
  - is the traditional operator;
  - is the operator many engineers feel more familiar with;
  - is used in the majority of the literature.
- The δ-operator, δ, has the advantages of:
  - emphasizing the link between continuous and discrete systems (resembles a differential);
  - δ-expressions converge to familiar continuous expressions as $\Delta \to 0$, which is intuitive;
  - is numerically vastly superior at fast sampling rates when properly implemented.

- Analysis of digital systems relies on discrete time versions of the continuous operators:
  - the discrete version of the differential operator is difference operator;
  - the discrete version of the Laplace Transform is either the Z-transform (associated with the shift operator) or the γ-transform (associated with the δ-operator).

- With the help of these operators,
  - continuous time differential equation models can be converted to discrete time difference equation models;
  - continuous time transfer or state space models can be converted to discrete time transfer or state space models in either the shift or δ operators.