ELEC ENG 4CL4:

Control System Design

Notes for Lecture #3

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Chapter 3

Modeling

Topics to be covered include:

- How to select the appropriate model complexity
- How to build models for a given plant
- How to describe model errors.
- How to linearize nonlinear models

It also provides a brief introduction to certain commonly used models, including

- State space models
- High order differential and high order difference equation models

The Raison *d'être* for Models

The basic idea of feedback is tremendously compelling. Recall the mould level control problem from Chapter 2. Actually, there are only three ways that a controller could manipulate the valve: open, close or leave it as it is. Nevertheless, we have seen already that the precise way this is done involves subtle trade-offs between conflicting objectives, such as speed of response and sensitivity to measurement noise.

The power of a mathematical model lies in the fact that it can be simulated in hypothetical situations, be subject to states that would be dangerous in reality, and it can be used as a basis for synthesizing controllers.

Model Complexity

In building a model, it is important to bear in mind that all real processes are complex and hence any attempt to build an exact description of the plant is usually an impossible goal. Fortunately, feedback is usually very forgiving and hence, in the context of control system design, one can usually get away with rather simple models, provided they capture the essential features of the problem.

We introduce several terms:

- **Nominal model**. This is an approximate description of the plant used for control system design.
- **Calibration model**. This is a more comprehensive description of the plant. It includes other features not used for control system design but which have a direct bearing on the achieved performance.
- **Model error**. This is the difference between the nominal model and the calibration model. Details of this error may be unknown but various bounds may be available for it.

Building Models

A first possible approach to building a plant model is to postulate a specific model structure and to use what is known as a *black box* approach to modeling. In this approach one varies, either by trial and error or by an algorithm, the model parameters until the dynamic behavior of model and plant match sufficiently well.

An alternative approach for dealing with the modeling problem is to use physical laws (such as conservation of mass, energy and momentum) to construct the model. In this approach one uses the fact that, in any real system, there are *basic phenomenological laws* which determine the relationships between all the signals in the system.

In practice, it is common to combine both black box and phenomenological ideas to building a model.

Control relevant models are often quite simple compared to the true process and usually combine physical reasoning with experimental data.

Types of plant models

Physical systems (plants) can typically be described as dynamical systems, and consequently they can be modeled using *differential equations* for continuous-time models and *difference equations* for discrete-time models.

Commonly used forms for modeling dynamical systems are:

- State space models (continuous or discrete time)
- Higher-order differential or difference equations

Simulating/solving models

A *linear* or *nonlinear* differential or difference equation can be simulated (i.e., solved numerically) in the time domain using ODE solvers, such as those available in Matlab.

An analytical solution can be obtained for a linear time-invariant (LTI) system.

Two powerful tool that can be used for LTI systems are:

1. the **Laplace transform**, which transforms a differential equation into an algebraic equation, and
2. the ***z*-transform**, which transforms a difference equation into an algebraic equation.

From this algebraic equation, a *transfer function* can be obtained, which provides important information about the response properties of the system, such as its frequency response and its stability.

Analyzing control systems

If the plant model is linear, and the other parts of the control loop can be implemented using linear components, then *transfer functions* for the *entire control loop* can be derived.

As we will see later in this course, these transfer functions offer a power tool for analysis and synthesis of control systems.

Consequently, for nonlinear systems we will typically attempt to derive an approximate linear model, i.e., *linearize* a nonlinear model.

State Space Models

For continuous time systems

$$\frac{dx}{dt} = f(x(t), u(t), t)$$
$$y(t) = g(x(t), u(t), t)$$

For discrete time systems

$$x[k+1] = f_d(x[k], u[k], k)$$
$$y[k] = g_d(x[k], u[k], k)$$

Linear State Space Models

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \mathbf{A}x(t) + \mathbf{B}u(t) \\
y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)
\end{aligned}
$$

Example 3.3

Consider the simple electrical network shown in *Figure 3.1*. Assume we want to model the voltage *v*(*t*)

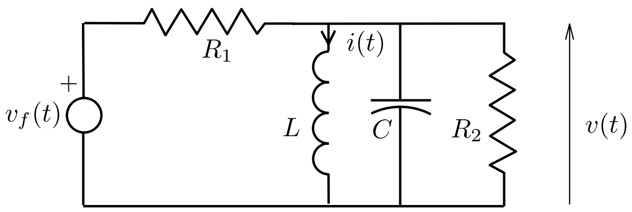


Figure 3.1: *Electrical network. State space model.*

On applying fundamental network laws we obtain the following equations:

$$v(t) = L\frac{di(t)}{dt}$$

$$\frac{v_f(t) - v(t)}{R_1} = i(t) + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$$

These equations can be rearranged as follows:

$$
\begin{aligned}
\frac{di(t)}{dt} &= \frac{1}{L}v(t) \\
\frac{dv(t)}{dt} &= -\frac{1}{C}i(t) - \left(\frac{1}{R_1C} + \frac{1}{R_2C}\right)v(t) + \frac{1}{R_1C}v_f(t)
\end{aligned}
$$

We have a linear state space model with

$$
\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{R_1C} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad \mathbf{D} = \mathbf{0}
$$

Example 3.4

Consider a separately excited d.c. motor. Let $v_a(t)$ denote the armature voltage, $\theta(t)$ the output angle. A simplied schematic diagram of this system is shown in Figure 3.2.

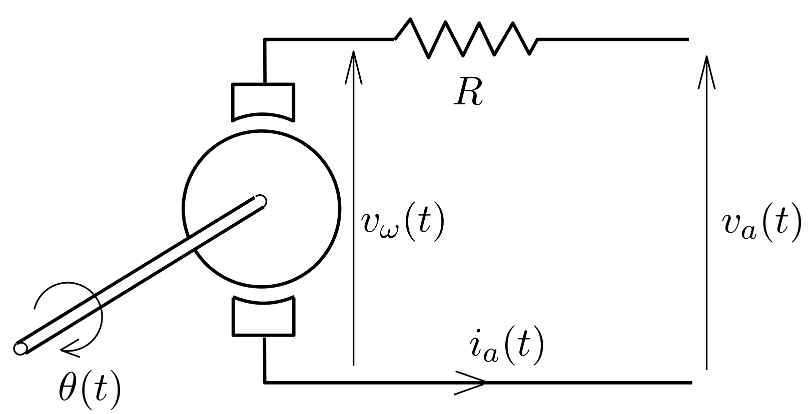
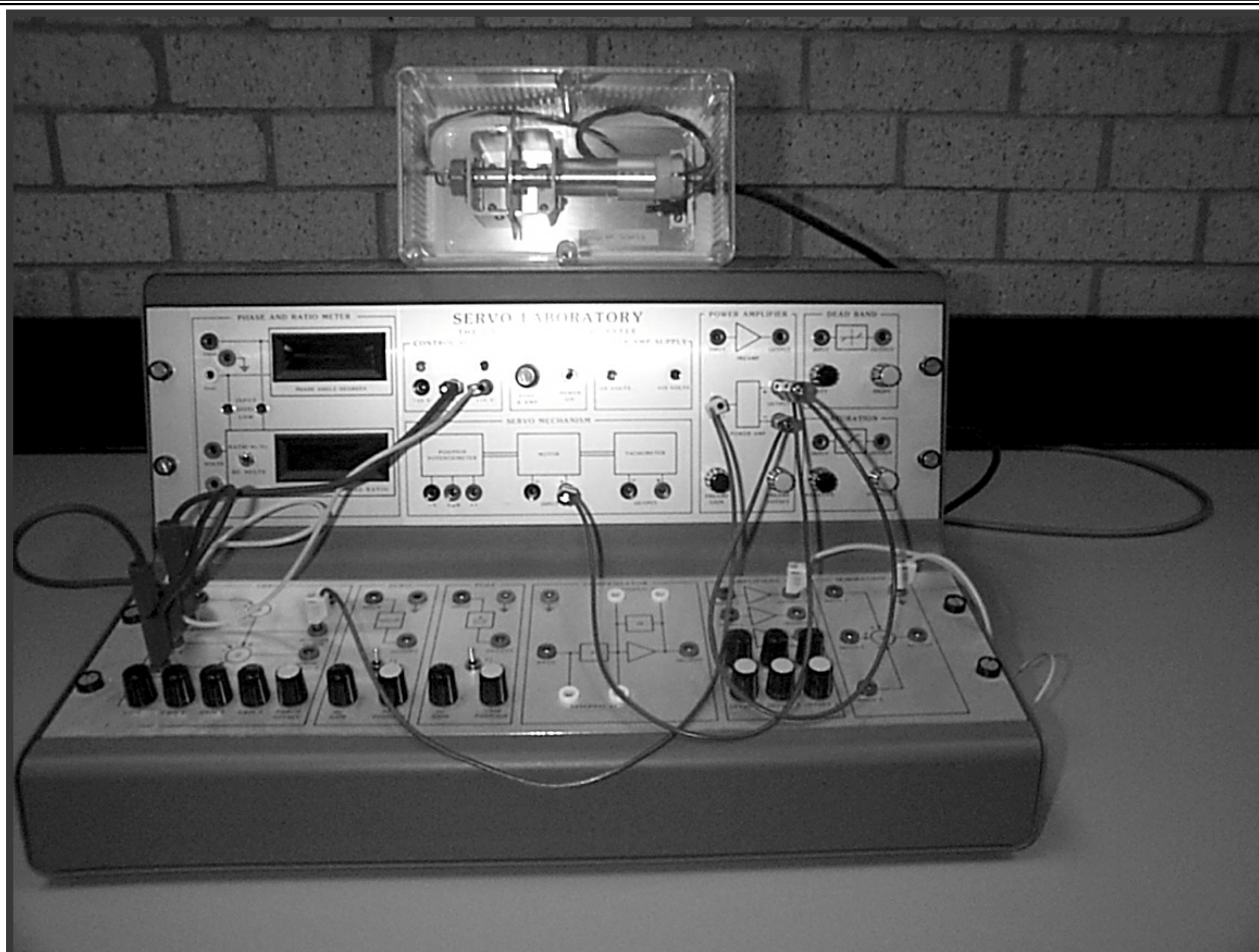
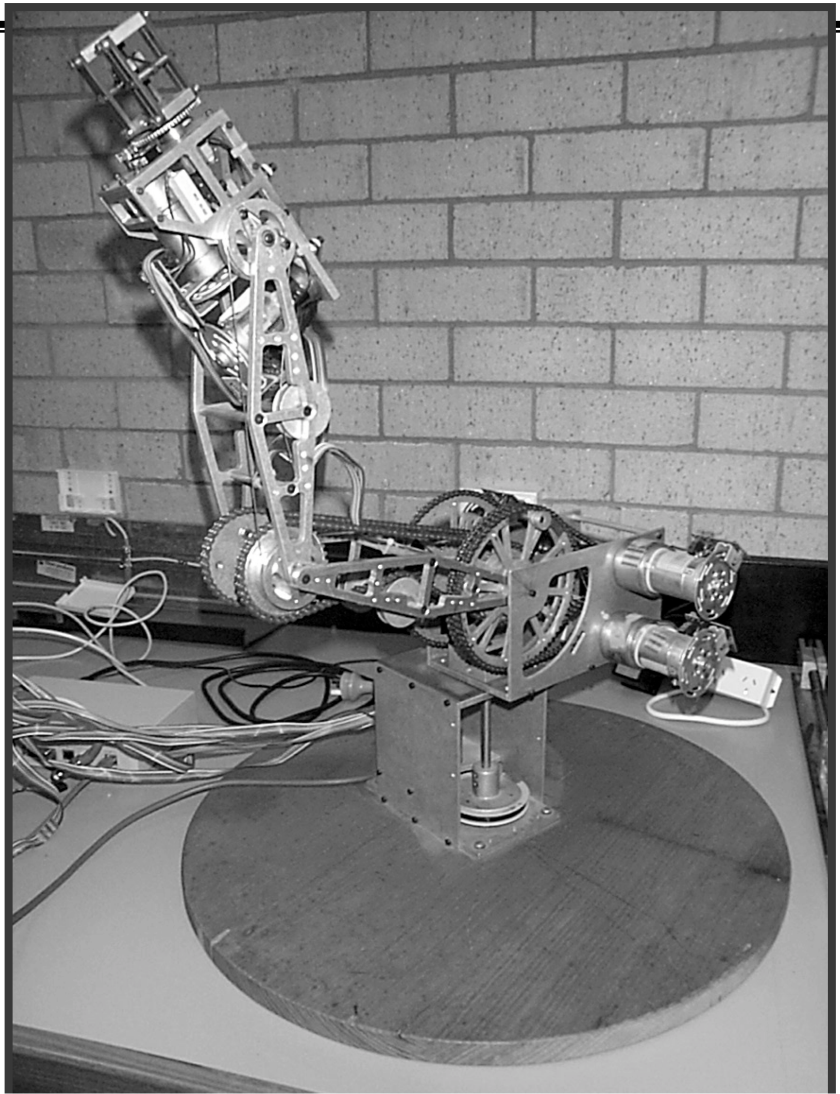


Figure 3.2: *Simplified model of a d.c. motor*

A laboratory servo kit



A demonstration robot containing several servo motors



Let

J	- be the inertia of the shaft
$\tau_e(t)$	- the electrical torque
$i_a(t)$	- the armature current
$k_1; k_2$	- constants
R	- the armature resistance

Application of well known principles of physics tells us that the various variables are related by:

$$J\ddot{\theta}(t) = \tau_e(t) = k_1 i_a(t)$$

$$v_\omega(t) = k_2\dot{\theta}(t)$$

$$i_a(t) = \frac{v_a(t) - k_2\dot{\theta}(t)}{R}$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-k_1 k_2}{R} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_1}{R} \end{bmatrix} v_a(t)$$

Solution of Continuous Time State Space Models

A key quantity in determining solutions to state equations is the *matrix exponential* defined as

$$e^{\mathbf{A}t} = \mathbf{I} + \sum_{i=1}^{\infty} \frac{1}{i!}\mathbf{A}^i t^i$$

The explicit solution to the linear state equation is then given by

$$x(t) = e^{\mathbf{A}(t-t_o)}x_o + \int_{t_o}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$