ELEC ENG 4CL4:

Control System Design

Notes for Lecture #5

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Chapter 4

Continuous Time Signals

Specific topics to be covered include:

- linear high order differential equation models
- Laplace transforms, which convert linear differential equations to algebraic equations, thus greatly simplifying their study
- methods for assessing the stability of linear dynamic systems
- frequency response.

Linear Continuous Time Models

The linear form of this model is:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_{n-1}\frac{d^{n-1}}{dt^{n-1}}u(t) + \ldots + b_0 u(t)$$

Introducing the Heaviside, or differential, operator $\rho\langle\circ\rangle$:

$$\rho\langle f(t)\rangle = \rho f(t) \triangleq \frac{df(t)}{dt}$$

$$\rho^n\langle f(t)\rangle = \rho^n f(t) = \rho\left\langle \rho^{n-1}\langle f(t)\rangle\right\rangle = \frac{df^n(t)}{dt^n}$$

We obtain:

$$\rho^n y(t) + a_{n-1}\rho^{n-1}y(t) + \ldots + a_0 y(t) = b_{n-1}\rho^{n-1}u(t) + \ldots + b_0 u(t)$$

Laplace Transforms

The study of differential equations of the type described above is a rich and interesting subject. Of all the methods available for studying linear differential equations, one particularly useful tool is provided by Laplace Transforms.

Definition of the Transform

Consider a continuous time signal $y(t)$; $0 \leq t < \infty$. The Laplace transform pair associated with $y(t)$ is defined as

$$\mathcal{L}\left[y(t)\right] = Y(s) = \int_{0^-}^{\infty} e^{-st} y(t) dt$$

$$\mathcal{L}^{-1}\left[y(s)\right] = y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} Y(s) ds$$

A key result concerns the transform of the derivative of a function:

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^-)$$

Table 4.1: *Laplace transform table*

$f(t) \quad (t \geq 0)$	$\mathcal{L}[f(t)]$	Region of Convergence
1	$\frac{1}{s}$	$\sigma > 0$
$\delta_D(t)$	1	$\lvert\sigma\rvert < \infty$
t	$\frac{1}{s^2}$	$\sigma > 0$
$t^n \quad n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$e^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{s-\alpha}$	$\sigma > \Re\{\alpha\}$
$te^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{(s-\alpha)^2}$	$\sigma > \Re\{\alpha\}$
$\cos(\omega_o t)$	$\frac{s}{s^2+\omega_o^2}$	$\sigma > 0$
$\sin(\omega_o t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\sigma > 0$
$e^{\alpha t}\sin(\omega_o t+\beta)$	$\frac{(\sin\beta)s+\omega_o^2\cos\beta-\alpha\sin\beta}{(s-\alpha)^2+\omega_o^2}$	$\sigma > \Re\{\alpha\}$
$t\sin(\omega_o t)$	$\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$t\cos(\omega_o t)$	$\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$\mu(t)-\mu(t-\tau)$	$\frac{1-e^{-s\tau}}{s}$	$\lvert\sigma\rvert < \infty$

Table 4.2: *Laplace transform properties. Note that* $F_i(s) = \quad [f_i(t)], Y(s) = \quad [y(t)], k \in \{1,2,3,\ldots\}, \ f_1(t) = f_2)(t) = 0 \ \forall t < 0.$

$f(t)$	$\mathcal{L}[f(t)]$	Names
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$	Linear combination
$\frac{dy(t)}{dt}$	$sY(s) - y(0^-)$	Derivative Law
$\frac{d^k y(t)}{dt^k}$	$s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left. \frac{d^{i-1} y(t)}{dt^{i-1}} \right\|_{t=0^-}$	High order derivative
$\int_{0^-}^{t} y(\tau) d\tau$	$\frac{1}{s} Y(s)$	Integral Law
$y(t-\tau)\mu(t-\tau)$	$e^{-s\tau} Y(s)$	Delay
$ty(t)$	$-\frac{dY(s)}{ds}$	
$t^k y(t)$	$(-1)^k \frac{d^k Y(s)}{ds^k}$	
$\int_{0^-}^{t} f_1(\tau) f_2(t-\tau) d\tau$	$F_1(s) F_2(s)$	Convolution
$\lim_{t \to \infty} y(t)$	$\lim_{s \to 0} sY(s)$	Final Value Theorem
$\lim_{t \to 0^+} y(t)$	$\lim_{s \to \infty} sY(s)$	Initial Value Theorem
$f_1(t) f_2(t)$	$\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F_1(\zeta) F_2(s-\zeta) d\zeta$	Time domain product
$e^{at} f_1(t)$	$F_1(s-a)$	Frequency Shift

Transfer Functions

Taking Laplace Transforms converts the differential equation into the following algebraic equation

$$s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \ldots + a_0 Y(s)$$

$$= b_{n-1} s^{n-1} U(s) + \ldots + b_0 U(s) + f(s, x_o)$$

This can be expressed as $Y(s) = G(s)U(s)$

where

$$G(s) = \frac{B(s)}{A(s)}$$

and

$$A(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$$

$$B(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0$$

G(*s*) is called the *transfer function*.

Transfer Functions for Continuous Time State Space Models

Taking Laplace transform in the state space model equations yields

$$sX(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s)$$

and hence

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}x(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$
$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0)$$

$$Y(s) = \mathbf{G}(s)U(s)$$
$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

G(*s*) is the system transfer function.

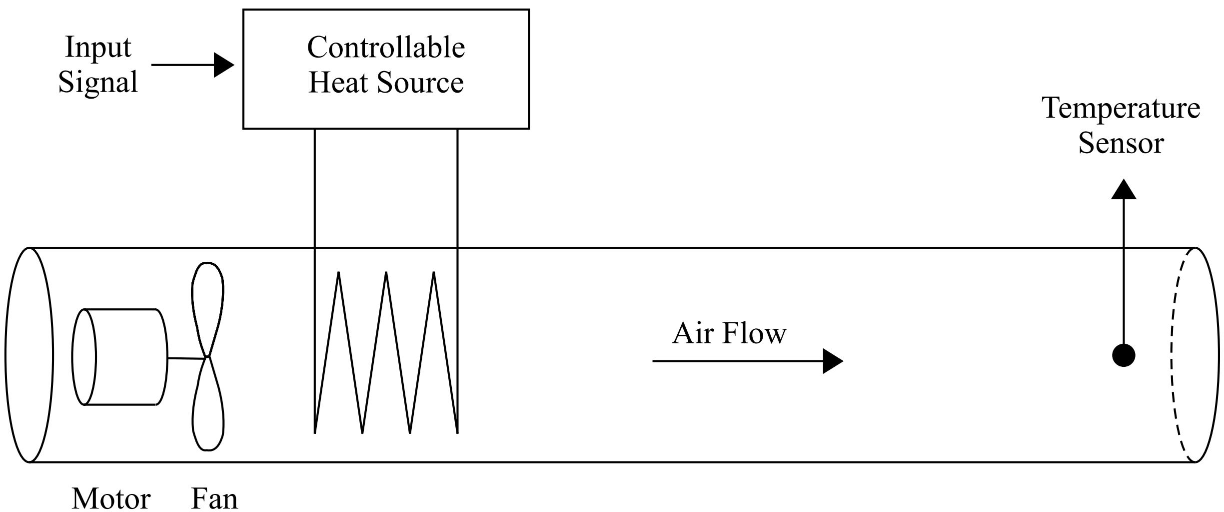
Often practical systems have a time delay between input and output. This is usually associated with the transport of material from one point to another. For example, if there is a conveyor belt or pipe connecting different parts of a plant, then this will invariably introduce a delay.

The transfer function of a pure delay is of the form (see Table 4.2):

$$H(s) = e^{-sT_d}$$

where T_d is the delay (in seconds). T_d will typically vary depending on the transportation speed.

Example 4.4 (Heating system). *As a simple example of a system having a pure time delay consider the heating system shown below.*



The transfer function from input (the voltage applied to the heating element) to the output (the temperature as seen by the thermocouple) is approximately of the form:

$$H(s) = \frac{Ke^{-sT_d}}{(\tau s + 1)}$$

Summary

Transfer functions describe the input-output properties of linear systems in algebraic form.

Stability of Transfer Functions

We say that a system is stable if any bounded input produces a bounded output for all bounded initial conditions. In particular, we can use a partial fraction expansion to decompose the total response of a system into the response of each pole taken separately. For continuous-time systems, we then see that stability requires that the poles have strictly negative real parts, i.e., they need to be in the open left half plane (OLHP) of the complex plane $\boxed{s}$. This implies that, for continuous time systems, the stability boundary is the imaginary axis.

Consider an individual pole of a system transfer function at $s = s_1 = \sigma_1 + j\Omega_1$:

$$H_1(s) = \frac{K}{s - s_1}.$$

The impulse response of this pole is:

$$\begin{aligned} h_1(t) &= \mathcal{L}^{-1}[H_1(s)] = \mathcal{L}^{-1}\left[\frac{K}{s - s_1}\right] \\ &= Ke^{s_1 t} = Ke^{\sigma_1 t}e^{j\Omega_1 t}. \end{aligned}$$

Note that if:

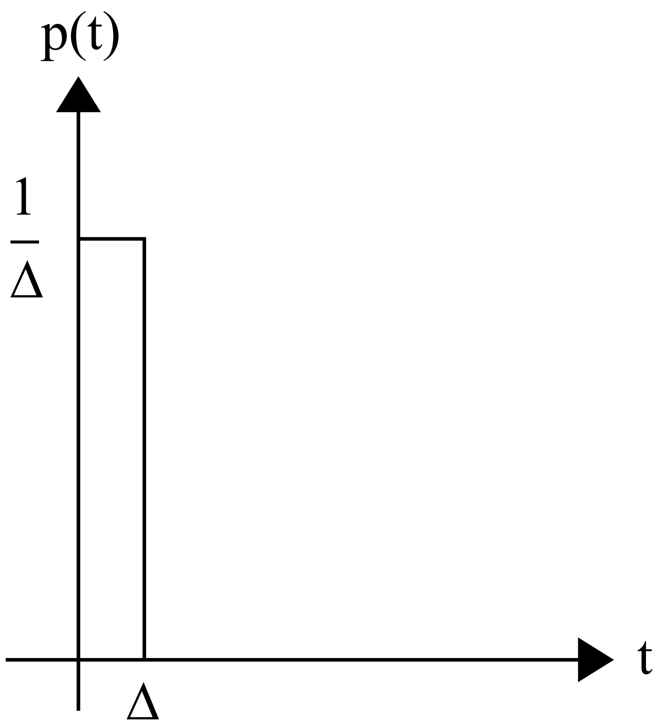
1. $\sigma_1 < 0$, then $e^{\sigma_1 t} \to 0$ as $t \to \infty \Rightarrow$ stable pole
2. $\sigma_1 > 0$, then $e^{\sigma_1 t} \to \infty$ as $t \to \infty \Rightarrow$ unstable pole
3. $\sigma_1 = 0$, then $e^{\sigma_1 t} = 1$ for $\forall t \Rightarrow$ unstable pole

Impulse and Step Responses of Continuous-Time Linear Systems

The transfer function of a continuous time system is the Laplace transform of its response to an impulse (Dirac's delta) with zero initial conditions.

The impulse function can be thought of as the limit ($\Delta \rightarrow 0$) *of the pulse shown on the next slide.*

Figure 4.2: *Discrete pulse*



Steady State Step Response

The steady state response (provided it exists) for a unit step is given by

$$\lim_{t\to\infty} y(t) = y_\infty = \lim_{s\to\infty} sG(s)\frac{1}{s} = G(0)$$

where $G(s)$ is the transfer function of the system.

We define the following indicators:

Steady state value, y_∞: the final value of the step response (this is meaningless if the system has poles in the CRHP).

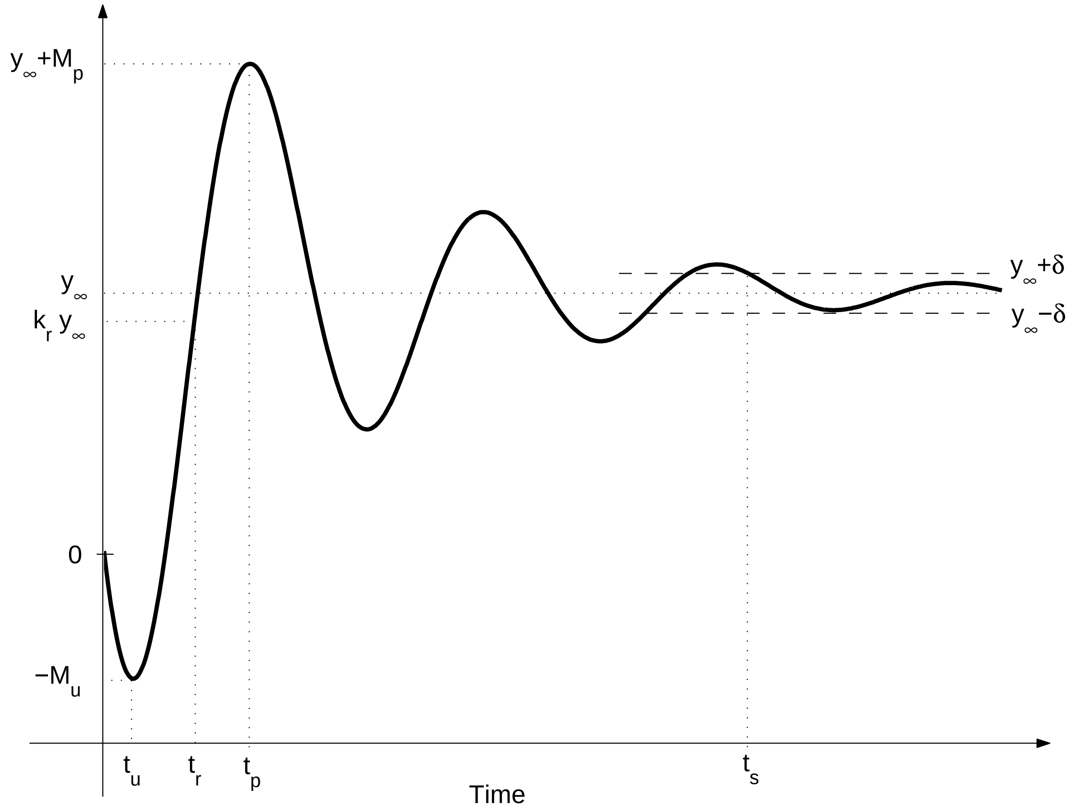
Rise time, t_r: The time elapsed up to the instant at which the step response reaches, for the first time, the value $k_r y_\infty$. The constant k_r varies from author to author, being usually either 0.9 or 1.

Overshoot, M_p: The maximum instantaneous amount by which the step response exceeds its final value. It is usually expressed as a percentage of y_∞

Undershoot, M_u: the (absolute value of the) maximum instantaneous amount by which the step response falls below zero.

Settling time, t_s: the time elapsed until the step response enters (without leaving it afterwards) a specified deviation band, $\pm\delta$, around the final value. This deviation δ, is usually defined as a percentage of y_∞, say 2% to 5%.

Figure 4.3: *Step response indicators*



Poles, Zeros and Time Responses

We will consider a general transfer function of the form

$$H(s) = K \frac{\prod_{i=1}^{m}(s-\beta_i)}{\prod_{l=1}^{n}(s-\alpha_l)}$$

β_1, β_1,…, β_m and α_1, α_2, ,,, α_n are the zeros and poles of the transfer function, respectively. The relative degree is $n_r \stackrel{\Delta}{=} n-m$.

Poles

Recall that any scalar rational transfer function can be expanded into a partial fraction expansion, each term of which contains either a single real pole, a complex conjugate pair or multiple combinations with repeated poles.

First Order Pole

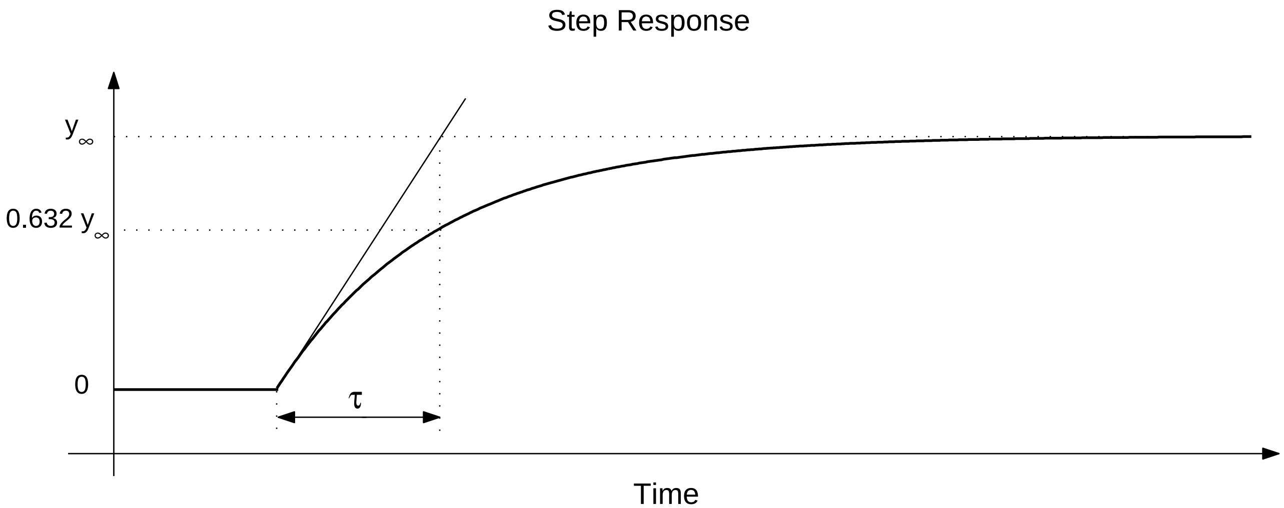
A general first order pole contributes

$$H_1(s) = \frac{K}{\tau s + 1}$$

The response of this system to a unit step can be computed as

$$y(t) = \mathcal{L}^{-1}\left[\frac{K}{s(\tau s + 1)}\right] = \mathcal{L}^{-1}\left[\frac{K}{s} - \frac{K\tau}{\tau s + 1}\right] = K(1 - e^{-\frac{t}{\tau}})$$

Figure 4.4: *Step response of a first order system*



A Complex Conjugate Pair

For the case of a pair of complex conjugate poles, it is customary to study a *canonical second order system* having the transfer function.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\psi\omega_n s + \omega_n^2}$$

Step Response for Canonical Second Order Transfer Function

$$
\begin{aligned}
Y(s) &= \frac{1}{s} - \frac{s+\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} - \frac{\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} \\
&= \frac{1}{s} - \frac{1}{\sqrt{1-\psi^2}}\left[\sqrt{1-\psi^2}\,\frac{s+\psi\omega_n}{(s+\psi\omega_n)^2+\omega_d^2} - \psi\,\frac{\omega_d}{(s+\psi\omega_n)^2+\omega_d^2}\right]
\end{aligned}
$$

On applying the inverse Laplace transform we finally obtain

$$
y(t) = 1 - \frac{e^{-\psi\omega_n t}}{\sqrt{1-\psi^2}}\sin(\omega_d t + \beta)
$$

Figure 4.5: *Pole location and unit step response of a canonical second order system.*