Electrical Engineering 4CL4: Control System Design

Solutions to Midterm Exam 2004

1. A *closed-loop* controller is preferable to an *open-loop* controller for most control problems.

a. Explain why this is the case.

b. List under what conditions an *open-loop* controller might be acceptable, and list the advantages of using an open-loop controller as compared to a closed-loop controller in such a case. (20 pts)

a. Closed-loop controllers are more forgiving in the presence of modelling errors, system instabilities, unknown initial conditions and disturbances to any signal in the system, because the *actual* output of the plant is being taken into account when producing the control signal.

b. An open-loop controller may be sufficient if:

   i. a very accurate model of the plant is known,

   ii. the model and its inverse are stable, and

   iii. disturbances and initial conditions are negligible.

   Some advantages of an open-loop controller are:

   i. no sensors required,

   ii. the controller may be reducible to a very simple system (e.g., an IIR filter), and

   iii. no transmission of sensor information required, and consequently a possible source of signal delay is removed.

2. The dynamics of a system are described by the differential equation:

$$\frac{d^3x(t)}{dt^3}+10\frac{d^2x(t)}{dt^2}+31\frac{dx(t)}{dt}+30x(t)=e^{-x(t)}.$$

a. Linearize this equation for $x(t)$ near 0.

b. Express the linearized equation from part a. in state-space form, assuming that the output of the state-space model is the variable $x(t)$. (20 pts)

a. The nonlinear term $e^{-x(t)}$ is linearized via the Taylor series approximation:

$$f(x)=e^{-x(t)}\approx f(x_0)+\left.\frac{df}{dx}\right|_{x(t)=x_0}\left(x(t)-x_0\right)=e^0-e^{-x(t)}\Big|_{x(t)=0}\left(x(t)-0\right)=1-x(t),$$

giving the linear equation:

$$\frac{d^3x(t)}{dt^3}+10\frac{d^2x(t)}{dt^2}+31\frac{dx(t)}{dt}+30x(t)=1-x(t)$$

$$\Rightarrow\frac{d^3x(t)}{dt^3}+10\frac{d^2x(t)}{dt^2}+31\frac{dx(t)}{dt}+31x(t)=1.$$

b. To obtain a state-space representation of the linearized model, we define three state variables $x_1(t)=x(t)$, $x_2(t)=\frac{dx(t)}{dt}$ and $x_3(t)=\frac{d^2x(t)}{dt^2}$, the input variable $u(t)=1$, and the output variable $y(t)=x(t)$, giving:

$$\frac{dx_1(t)}{dt}=x_2(t),$$

$$\frac{dx_2(t)}{dt}=x_3(t),$$

$$\frac{dx_3(t)}{dt}=-31x_1(t)-31x_2(t)-10x_3(t)+u(t),$$

$$y(t)=x_1(t).$$

3. For a one-d.o.f., unity-feedback control system with the controller and plant transfer functions:

$$C(s) = \frac{K(s+6)}{s(s+1)} \qquad \textbf{and} \qquad G_o(s) = \frac{1}{s+3},$$

determine the range of values for K that ensures stability. (20 pts)

This control loop has the characteristic polynomial:

$$\begin{aligned} A_{cl}(s) &= A_o(s)L(s) + B_o(s)P(s) \\ &= s(s+1)(s+3) + K(s+6) \\ &= s^3 + 4s^2 + (3+K)s + 6K. \end{aligned}$$

Evaluating Routh's array:

$$\begin{array}{lll} s^3 & 1 & 3+K \\ s^2 & 4 & 6K \\ s^1 & 3-\frac{K}{2} & \\ s^0 & 6K & \end{array}$$

gives two conditions for stability:

$$3 - \frac{K}{2} > 0 \Rightarrow K < 6 \qquad \text{and} \qquad 6K > 0 \Rightarrow K > 0.$$

Combining the criteria gives $0 < K < 6$.

4. **Consider a one-d.o.f., unity-feedback control system with the controller and plant transfer functions:**

$$C(s) = 5 \qquad \textbf{and} \qquad G_o(s) = \frac{1}{(s+3)(s+6)}.$$

If the system has the input disturbance $d_i(t) = 5\sin(6t)$, what does this input disturbance contribute to the plant output $y(t)$ in the steady state? (20 pts)

In a one-d.o.f., unity-feedback nominal control loop, the plant output is related to an input disturbance according to the nominal input disturbance sensitivity:

$$Y(s) = S_{io}(s) D_i(s).$$

For the given control and plant transfer functions, the nominal input disturbance sensitivity is:

$$S_{io}(s) = \frac{G_o(s)}{1+G_o(s)C(s)} = \frac{B_o(s)L(s)}{A_o(s)L(s)+B_o(s)P(s)}$$
$$= \frac{1}{(s+3)(s+6)+5} = \frac{1}{s^2+9s+23}.$$

For a sinusoidal disturbance, the steady-state response can be found by evaluating the frequency response of the nominal input disturbance sensitivity:

$$S_{io}(j\omega) = \frac{1}{-\omega^2 + j9\omega + 23},$$

at the frequency of the sinusoid $\omega = 6$:

$$S_{io}(j6) = \frac{1}{6^2 + j9\cdot 6 + 23}$$
$$= \frac{1}{-36 + j54 + 23}$$
$$= \frac{1}{-13 + j54}$$
$$= -0.0042 - j0.0175,$$

giving:

$$y(t) = \left|S_{io}(j6)\right| 5\sin\left(6t + \angle S_{io}(j6)\right)$$
$$= 0.018 \cdot 5\sin(6t - 1.807)$$
$$= 0.09\sin(6t - 1.807).$$

5. A system has the transfer function:

$$H(s) = \frac{-2s+1}{s^2+2s+1}.$$

a. Will the *step response* of this system exhibit undershoot or overshoot? Explain your answer in terms of the locations of the poles and zeros of the transfer function.

b. If the step response does exhibit undershoot or overshoot, find both the *magnitude* and the *time* of maximum undershoot or overshoot. (20 pts)

a. This transfer function has two real-valued poles at $s=-1$, which will not contribute directly to overshoot or undershoot. (A pair of complex conjugate poles could contribute to overshoot.) The transfer function has one nonminimum-phase zero at $s=\frac{1}{2}$. *Minimum-phase* zeros can produce overshoot; conversely, *nonminimum-phase* zeros produce undershoot. The closer the zero is to the imaginary axis, i.e., the smaller the magnitude of the zero, the larger the contribution to overshoot or undershoot. We note that the nonminimum-phase zero at $s=\frac{1}{2}$ is closer to the imaginary axis than the poles at $s=-1$, so we would expect that the system would exhibit substantial *undershoot*.

a. The Laplace transform of the step response $y(t)$ of this system is:

$$Y(s) = H(s)U(s) = H(s)\frac{1}{s} = \frac{-2s+1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{3}{(s+1)^2}.$$

Taking the inverse Laplace transform of $Y(s)$ gives:

$$y(t) = 1 - e^{-t} - 3te^{-t}, \quad t \geq 0.$$

We find the minimum of $y(t)$ by setting slope of $y(t)$ equal to zero:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = e^{-t} - 3e^{-t} + 3te^{-t} = 0.$$

One solutions is at $t \to \infty$, which corresponds to the steady-state response $y_\infty = 1$, not the minimum. The second solution, which corresponds to the minimum, can be found as follows:

$$\Rightarrow e^{-t} - 3e^{-t} + 3te^{-t} = 0$$
$$\Rightarrow 3te^{-t} = 2e^{-t}$$
$$\Rightarrow 3t = 2$$
$$\Rightarrow t = \tfrac{2}{3},$$

giving a minimum value of $y\left(\frac{2}{3}\right) = 1 - e^{-\frac{2}{3}} - 3\cdot\frac{2}{3}\cdot e^{-\frac{2}{3}} \approx -0.5403$.

Consequently, the undershoot is $M_u = \left|y\left(\frac{2}{3}\right)\right| = 0.5403$, occurring at $t=\frac{2}{3}$.

6. The Robust Stability Theorem, as given on page 10 of this exam paper, describes a sufficient condition for the stability of a true feedback control loop in the presence of a plant modeling error. In some situations, the model used for the controller in the design process is somewhat different from the controller than is actually implemented in the true feedback control loop. (For example, a digital PID controller may be used to approximate the behaviour of an analog PID controller that was used in the design process.) Such differences may be considered controller modeling errors.

Describe how you might adapt the Robust Stability Theorem so as to give a sufficient condition for the stability of a true feedback control loop in the presence of a controller modeling error instead of a plant modeling error. (20 pts)

First we note that the Robust Stability Theorem depends on the open-loop transfer function $G_o(s)C(s)$, i.e., the product of the controller and plant transfer functions, and that neither of these transfer function appears alone in Eqn. (5.9.6). Consequently, it does not matter if the error is attributed to the plant, the controller or both.

Defining the nominal and actual controller models as $C_o(s)$ and $C(s)$, respectively, gives the multiplicative controller modeling error:

$$C_\Delta(s) \triangleq \frac{C(s) - C_o(s)}{C_o(s)}.$$

Defining the nominal complementary sensitivity for the actual plant model $G(s)$ and the nominal controller model $C_o(s)$:

$$T_{co}(s) \triangleq \frac{G(s)C_o(s)}{1 + G(s)C_o(s)},$$

we can now give the Robust Stability Theorem in the presence of a controller modelling error as:

$$|T_{co}(j\omega)||C_\Delta(j\omega)| = \left|\frac{G(j\omega)C_o(j\omega)}{1 + G(j\omega)C_o(j\omega)}\right||C_\Delta(j\omega)| < 1, \quad \forall \omega.$$