Problem 4.4, p 110

$$\frac{dy(t)}{dt} + y(t)(1 - 0.2y^2(t)) = 2u(t) \quad \text{input-output model}$$

4.4.1 Determine the transfer function for the linearized model as a function of (Y_Q, u_Q):

At (Y_Q, u_Q), $Y_Q = 0.2Y_Q^3 + 2u_Q$

Simplify $-0.2y^3(t)$ term using Taylor's series expansion:

$$-0.2y^3(t) \simeq -0.2Y_Q^3 - 0.6Y_Q^2 \underbrace{(y(t) - Y_Q)}_{\Delta y(t)}$$

$$\frac{dy(t)}{dt} + y(t) - 0.2y^3(t) = 2u(t)$$

$$\frac{d(\Delta y(t) + \cancel{Y_Q})}{dt} + \Delta y(t) + \overbrace{Y_Q - 0.2Y_Q^3}^{= 2u_Q} - 0.6Y_Q^2\Delta y(t) = 2u(t)$$

$$2\Delta u(t) = 2(u(t) - u_Q)$$

$$\frac{d\Delta y(t)}{dt} + (1 - 0.6Y_Q^2)\Delta y(t) = 2\Delta u(t)$$

OR

Linearize $\frac{dy(t)}{dt}$ using 2D Taylor Series expansion:

$$f(x,z) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (z-b)\frac{\partial}{\partial z}\right] f \Big|_{(a,b)} + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial z}\right]^2 f \Big|_{(a,b)} + \dots$$

$$\frac{dy(t)}{dt} = \frac{d\Delta y(t)}{dt} \simeq 2u_Q - Y_Q + 0.2Y_Q^3 + \Delta u(t)2 + \Delta y(-1 + 0.6Y_Q^2)$$

$$= \cancel{Y_Q} - \cancel{0.2Y_Q^3} + \cancel{0.2Y_Q^3} - \cancel{Y_Q} + 2\Delta u(t) + (0.6Y_Q^2 - 1)\Delta y(t)$$

$$= 2u(t) + (0.6Y_Q^2 - 1)\Delta y(t)$$

$$\downarrow \mathcal{L}$$

$$s\Delta Y(s) = 2\Delta U(s) + (0.6Y_Q^2 - 1)\Delta Y(s)$$

$$\frac{(s - 0.6Y_Q^2 + 1)}{2}\frac{\Delta Y(s)}{\Delta U(s)} = 1$$

$$H_\Delta(s) = \frac{\Delta Y(s)}{\Delta U(s)} = \frac{2}{s + 1 - 0.6Y_Q^2}$$

4.4.2 $H_\Delta(s)$ is unstable if $1 - 0.6Y_Q^2 \leq 0$

$1.67 \leq Y_Q^2$ a.s. $Y_Q = 2$ is unstable

Problem 4.9 p111

$u(t) = 2\cos(0.5t)$

$U(s) = \dfrac{2s}{s^2+0.5^2}$

sol'n i)

$$U(s) \longrightarrow \boxed{H(s)} \xrightarrow{Y(s)}$$

$H(s) = \dfrac{-s+4}{s^2+5s+6}$

$$Y(s) = U(s)H(s)$$

$$= \frac{2s}{s^2+0.5^2} \cdot \frac{-s+4}{s^2+5s+6} = \frac{-2s^2+8s}{(s^2+0.5^2)(s+3)(s+2)}$$

$$= \frac{As+B}{(s^2+0.5^2)} + \frac{C}{s+3} + \frac{D}{s+2}$$

$$= \frac{(As+B)(s+3)(s+2) + C(s^2+0.5^2)(s+2) + D(s^2+0.5^2)(s+3)}{(s^2+0.5^2)(s+3)(s+2)}$$

$$= \frac{(As^2+Bs+3As+3B)(s+2) + C(s^3+2s^2+0.5^2s+0.5) + D(s^3+3s^2+0.5^2s+0.75)}{(s^2+0.5^2)(s+3)(s+2)}$$

$$= \frac{As^3+Bs^2+3As^2+3Bs+2As^2+2Bs+6As+6B+Cs^3+2Cs^2+C0.5^2s + 0.5C + Ds^3+3Ds^2+0.5^2Ds+0.75D}{(s^2+0.5^2)(s+3)(s+2)}$$

$$= \frac{(A+C+D)s^3+(B+3A+2A+2C+3D)s^2+(3B+2B+6A+C0.5^2+0.5^2D)s+(6B+0.5C+0.75D)}{(s^2+0.5^2)(s+3)(s+2)}$$

$s^3:\ A+C+D=0$

$s^2:\ B+5A+2C+3D=-2$

$s^1:\ 5B+6A+0.25C+0.25D=8$

$s^0:\ 6B+0.5C+0.75D=0$

$A=1.11$

$B=0.33$

$C=4.54$

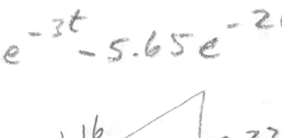
$D=-5.65$

MATLAB

$$\rightarrow Y(s) = \frac{1.11s+0.33}{s^2+0.5^2} + \frac{4.54}{s+3} - \frac{5.65}{s+2}$$

$$y(t) = 1.11\cos(0.5t) - 0.33\sin(0.5t) + 4.54e^{-3t} - 5.65e^{-2t}$$

$\downarrow t \to \infty$



$$y_\infty(t) = 1.11\cos(0.5t) - 0.33\sin(0.5t)$$

$$= 1.16\left(\cos(16.6^\circ)\cos(0.5t) - \sin(16.6^\circ)\sin(0.5t)\right)$$

$$= 1.16\cos(0.5t+0.29)$$

$A\cos(\omega t+\theta)$

$= A[\cos\theta\cos(\omega t) + \sin\theta\sin(\omega t)]$

Numbers are off

Problem 4.9 p111

$s = j\omega$

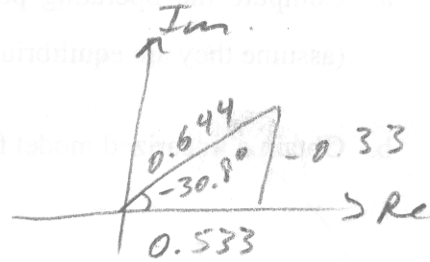
4 ii) $H(0.5j) = \dfrac{-(0.5j)+4}{(0.5j)^2+5(0.5j)+6}$

$= \dfrac{4-0.5j}{-0.25+2.5j+6} = \dfrac{(4-0.5j)(5.75-2.5j)}{(5.75+2.5j)(5.75-2.5j)}$

$= \dfrac{23-10j-2.875j-1.25}{33.06+6.25}$

$= \dfrac{21.75-12.875j}{39.31} = 0.553-0.33j$

$= 0.644 \angle -30.8^\circ$



$U(0.5j) = 2 \angle 0^\circ$

$Y(0.5j) = U(0.5j)\,H(0.5j) = 2(0.644) \angle -30.8^\circ + 0^\circ$

$= 1.288 \angle \underbrace{-30.8^\circ}_{-0.538 \text{ rad}}$

$y_\infty(t) = 1.288 \cos(0.5t - 0.538)$

i.e. $y_\infty(t) = 2|H(0.5j)| \cos(0.5t + \angle H(0.5j))$

4.20

step response: $y(t) = 1 - 0.5e^{-t} - 0.5e^{-2t}$

what is the transfer fcn?

$$U(s) = \frac{1}{s}$$

$$H(s) = \frac{Y(s)}{U(s)} = sY(s)$$

$$= s\mathcal{L}\left\{1 - 0.5e^{-t} - 0.5e^{-2t}\right\}$$

$$= s\left[\frac{1}{s} - \frac{0.5}{s+1} - \frac{0.5}{s+2}\right]$$

$$= \cancel{s}\left[\frac{(s+1)(s+2) - 0.5s(s+2) - 0.5s(s+1)}{s(s+1)(s+2)}\right]$$

$$= \frac{\cancel{s^2} + 3s + 2 - \cancel{0.5s^2} - \cancel{s} - \cancel{0.5s^2} - 0.5s}{(s+1)(s+2)}$$

$$= \frac{1.5s + 2}{(s+1)(s+2)} = \frac{3s + 4}{2(s+1)(s+2)}$$

4.14

i) $\omega_{n_0} = \omega_n$, $\zeta \neq \zeta_0$

$$\underset{\uparrow \atop \text{MME}}{G_\Delta(s)} = \frac{\dfrac{\cancel{F(s)}\,\omega_{n_0}^2}{s^2+2\zeta\omega_{n_0}s+\omega_{n_0}^2} - \dfrac{\cancel{F(s)}\,\omega_{n_0}^2}{s^2+2\zeta_0\omega_{n_0}s+\omega_{n_0}^2}}{\dfrac{\cancel{F(s)}\,\omega_{n_0}^2}{s^2+2\zeta_0\omega_{n_0}s+\omega_{n_0}^2}} = \frac{\overset{\text{AME}}{G_E(s)}}{G_0(s)}$$

$$= \frac{s^2+2\zeta_0\omega_{n_0}s+\omega_{n_0}^2}{s^2+2\zeta\omega_{n_0}s+\omega_{n_0}^2} - 1$$

$$= \frac{\cancel{s^2}+2\zeta_0 s+\cancel{\omega_{n_0}^2}-\cancel{s^2}-2\zeta\omega_{n_0}s-\cancel{\omega_{n_0}^2}}{s^2+2\zeta\omega_{n_0}s+\omega_{n_0}^2}$$

$$= \frac{2s(\zeta_0-\zeta)}{s^2+2\zeta\omega_{n_0}s+\omega_{n_0}^2}$$

$$G_\Delta(j\omega) = \frac{2j\omega(\zeta_0-\zeta)}{-\omega^2+2\zeta\omega_n j\omega+\omega_{n_0}^2} \quad \text{Freq. Response}$$

ii) $\zeta = \zeta_0$, $\omega_{n_0} \neq \omega_n$

$$G_\Delta(s) = \frac{\dfrac{F(s)\,\omega_n^2}{s^2+2\zeta_0\omega_n s+\omega_n^2} - \dfrac{\cancel{F(s)}\,\omega_{n_0}^2}{s^2+2\zeta_0\omega_{n_0}s+\omega_{n_0}^2}}{\dfrac{\cancel{F(s)}\,\omega_{n_0}^2}{s^2+2\zeta_0\omega_{n_0}s+\omega_{n_0}^2}} = \frac{G_E(s)}{G_0(s)}$$

$$= \frac{\omega_n^2(s^2+2\zeta_0\omega_{n_0}s+\omega_{n_0}^2)}{\omega_{n_0}^2(s^2+2\zeta_0\omega_n s+\omega_n^2)} - 1$$

$$= \frac{s^2(\omega_n^2-\omega_{n_0}^2)+2\zeta_0\omega_n\omega_{n_0}s(\omega_n-\omega_{n_0})}{\omega_{n_0}^2(s^2+2\zeta_0\omega_n s+\omega_n^2)}$$

$$G_\Delta(j\omega) = \frac{-\omega^2(\omega_n^2-\omega_{n_0}^2)+2j\omega\zeta_0\omega_n\omega_{n_0}(\omega_n-\omega_{n_0})}{\omega_{n_0}^2(-\omega^2+2j\omega\zeta_0\omega_n+\omega_n^2)}$$