Chapter 9 - Solved Problems

Solved Problem 9.1. *Consider an internally stable feedback loop with*

$$S_o(s) = \frac{s(s+2)}{s^2+4s+2} \tag{1}$$

Determine whether Lemma 9.1 or Lemma 9.2 of the book applies to this system.

Solutions to Solved Problem 9.1

Solved Problem 9.2. *Consider a linear plant having a nominal model having relative degree 2 and no poles in the open RHP.*

Compute a lower bound for the sensitivity peak using Lemma 9.1 from the book and assuming that the following specifications are met:

- $|S_o(j\omega)| < \epsilon$ *for* $0 \leq \omega \leq \omega_\ell$
- $|T_o(j\omega)| < \frac{\epsilon\,\omega_h^2}{\omega^2}$ *for* $\omega \geq \omega_h > \omega_\ell$

where $\epsilon = 0.1$, $\omega_\ell = 2$ $[rad/s]$ *and* $\omega_h = 3$ $[rad/s]$.

Solutions to Solved Problem 9.2

Solved Problem 9.3. *Consider a plant having a linear nominal model of relative degree 2 with no poles in the open RHP. Assume that the nominal model has a zero at* $s = c = 0.5$.

Further assume that the following design specifications are met:

S.1 $|S_o(j\omega)| < \epsilon\frac{\omega^2}{\omega_\ell^2}$ *for* $0 \leq \omega \leq \omega_\ell$

S.2 $|T_o(j\omega)| < \frac{\epsilon\,\omega_h^2}{\omega^2}$ *for* $\omega \geq \omega_h > \omega_\ell$

where $\epsilon = 0.1$, $\omega_\ell = 2$ $[rad/s]$ *and* $\omega_h = 3$ $[rad/s]$.

Find a lower bound for the complementary sensitivity peak.

Solutions to Solved Problem 9.3

Solved Problem 9.4. *Consider a stable plant with NMP zeros located at* $s = 1 \pm j\alpha$. *Further assume that we require* $|S_o(j\omega)| \leq \epsilon < 1$ *for all* $\omega \in [0, \omega_\ell]$. *Using Lemma 9.5 from the book investigate the effect of the imaginary part,* α *of the NMP zeros on the lower bound for the sensitivity peak.*

Solutions to Solved Problem 9.4

Solved Problem 9.5. *Consider a linear system having the nominal model*

$$G_o(s) = \frac{4(-s+4)}{(-s+2)(s+5)(s+2)} \tag{2}$$

Determine a lower limit for the sensitivity peak, assuming that we require that

$$|S_o(j\omega)| \leq 0.1 \quad \omega \in [0,\ 1] \tag{3}$$

$$|T_o(j\omega)| \leq 0.1 \quad \omega \in [8,\ \infty] \tag{4}$$

Solutions to Solved Problem 9.5

Chapter 9 - Solutions to Solved Problems

Solution 9.1. *Since the feedback loop is internally stable, no unstable pole-zero cancellation can occur. Thus, all open loop poles in the RHP will appear in the sensitivity numerator. Since no zero of* $S_o(s)$ *is located in the open RHP, then Lemma 9.1 applies.*

We can calculate the open loop transfer function $H_{ol}(s)$ *as*

$$H_{ol}(s) = S_o(s)^{-1} - 1 = \frac{2(s+1)}{s(s+2)} \tag{5}$$

Note that $H_{ol}(s)$ *has relative degree,* n_r*, equal to 1 and that*

$$\kappa = \lim_{s\to\infty} H_{ol}(s) = 2 \tag{6}$$

We can thus apply equation (9.2.3) from the book to obtain

$$\int_0^\infty \ln|S_o(j\omega)|\, d\omega = -\kappa\frac{\pi}{2} = -\pi \tag{7}$$

Solution 9.2. *We use (7), splitting the integration interval as follows:* $[0;\infty] = [0;\omega_\ell]\cup(\omega_\ell;\omega_h]\cup(\omega_h,\infty)$*. Then*

$$0 = \int_0^\infty \ln|S_o(j\omega)|\, d\omega = \int_0^{\omega_\ell} \ln|S_o(j\omega)|\, d\omega + \int_{\omega_\ell}^{\omega_h} \ln|S_o(j\omega)|\, d\omega + \int_{\omega_h}^{\infty} \ln|S_o(j\omega)|\, d\omega \tag{8}$$

$$< \int_0^{\omega_\ell} \ln\epsilon\, d\omega + \int_{\omega_\ell}^{\omega_h} \ln|S_{max}|\, d\omega + \int_{\omega_h}^{\infty} \ln\left|1 + \frac{\epsilon\,\omega_h^2}{\omega^2}\right| d\omega \tag{9}$$

The last integral on the right is[1]

$$\int_{\omega_h}^{\infty} \ln\left|1 + \frac{\epsilon\,\omega_h^2}{\omega^2}\right| d\omega = \int_{\omega_h}^{\infty} \ln\left(1 + \frac{\epsilon\,\omega_h^2}{\omega^2}\right) d\omega = 2\omega_h\sqrt{\epsilon}\arctan\sqrt{\epsilon} - \omega_h\ln(1+\epsilon) \stackrel{\triangle}{=} \omega_h f(\epsilon) \leq \omega_h\epsilon \tag{10}$$

From the above equations we conclude that

$$\ln|S_{max}| > \frac{\omega_\ell|\ln\epsilon| - \omega_h\epsilon}{\omega_h - \omega_\ell} \approx 4.4 \tag{11}$$

Note that the lower limit of the sensitivity peak increases as $\epsilon \to 0$ *and/or* $\omega_\ell \to \omega_h$

Solution 9.3. *To compute the lower bound for the complementary sensitivity peak, we can use Lemma 9.4 of the book (with* $\tau = 0$*) splitting the integration interval, as in Solved Problem 9.2.*

Note that specification [S.1] implies that the open loop transfer function $H_{ol}(s) = G_o(s)C(s)$ *has one or more poles at the origin. Thus equation (9.3.2) of the book is satisfied. Actually, S.1 would seem to imply that when* k_v *is computed from (9.3.4) the result is* $k_v = \infty$*.*

[1] To compute this integral use either a mathematical symbolic package such as MAPLE or MATHEMATICA, or the MATLAB symbolic toolbox, which is a MAPLE subset.

Thus

$$0 = \int_0^\infty \frac{\ln|T_o(j\omega)|}{\omega^2}\,d\omega = \int_0^{\omega_\ell} \frac{\ln|T_o(j\omega)|}{\omega^2}\,d\omega + \int_{\omega_\ell}^{\omega_h} \frac{\ln|T_o(j\omega)|}{\omega^2}\,d\omega + \int_{\omega_h}^{\infty} \frac{\ln|T_o(j\omega)|}{\omega^2}\,d\omega \tag{12}$$

$$< \underbrace{\int_0^{\omega_\ell} \frac{1}{\omega^2}\ln\left(1+\epsilon\frac{\omega^2}{\omega_\ell^2}\right)d\omega}_{I_1} + \ln|T_{max}|\underbrace{\int_{\omega_\ell}^{\omega_h}\frac{1}{\omega^2}\,d\omega}_{I_2} + \underbrace{\int_{\omega_h}^{\infty}\frac{1}{\omega^2}\ln\left(\frac{\epsilon\,\omega_h^2}{\omega^2}\right)d\omega}_{I_3} \tag{13}$$

These integrals can be computed using, for instance, MAPLE. This yields

$$I_1 = \frac{-\ln(1+\epsilon) + 2\sqrt{\epsilon}\arctan\sqrt{\epsilon}}{\omega_\ell} \tag{14}$$

$$I_2 = \frac{\omega_h - \omega_\ell}{\omega_h\omega_\ell} \tag{15}$$

$$I_3 = \frac{\ln(\epsilon) - 2}{\omega_h} \tag{16}$$

We can now use equation (9.3.11) of the book with $\tau = 0$, $M = 1$ *and* $k_v = \infty$. *Hence*

$$\ln|T_{max}| > \frac{\pi\omega_h\omega_\ell}{c(\omega_h - \omega_\ell)} + \frac{(\ln(1+\epsilon) - 2\sqrt{\epsilon}\arctan\sqrt{\epsilon})\,\omega_h}{\omega_h - \omega_\ell} + \frac{(2 + |\ln(\epsilon)|)\,\omega_\ell}{\omega_h - \omega_\ell} \approx 46.4 \tag{17}$$

It is interesting to note the factors which will cause this bound to grow. They are

- $\dfrac{\omega_\ell}{\omega_h}$ *tends to one, i.e. the design specification demands a very sharp transition from the pass band to the stop band in* $T_o(j\omega)$.
- *The parameter* ϵ *tends to zero. This means that the design specification requires an excessively flat frequency response.*
- *The ratio* $\dfrac{c}{\omega_\ell}$ *tends to zero. This is in agreement with time domain analysis previously given in Chapter 8 of the book.*

It is also interesting to note that the most significant contribution to this lower limit is the NMP zero. It accounts for roughly 82% of the limit.

Solution 9.4. *We apply Lemma 9.5 from the book, with* $M = 2$, $c_1 = 1 + j\alpha$ *and* $c_2 = 1 - j\alpha$, *and the right hand side in equation (9.4.2) of the book equal to 0 (since there are no unstable poles). Thus, for* c_1

$$\int_{-\infty}^{\infty} \ln|S_o(j\omega)|\frac{1}{1+(\alpha-\omega)^2}\,d\omega = \int_{-\infty}^{-\omega_\ell} \ln|S_o(j\omega)|\frac{1}{1+(\alpha-\omega)^2}\,d\omega +$$

$$+ \int_{-\omega_\ell}^{\omega_\ell} \ln|S_o(j\omega)|\frac{1}{1+(\alpha-\omega)^2}\,d\omega + \int_{\omega_\ell}^{\infty} \ln|S_o(j\omega)|\frac{1}{1+(\alpha-\omega)^2}\,d\omega \tag{18}$$

We can now substitute $\ln|S_o(j\omega)|$ *by its upper bound on every interval*

$$\max|S_o(j\omega)| = \begin{cases} S_{max} & \omega \in [-\infty,\ -\omega_\ell] \\ \epsilon & \omega \in [-\omega_\ell,\ \omega_\ell] \\ S_{max} & \omega \in [\omega_\ell,\ \infty] \end{cases} \tag{19}$$

Then

$$\int_{-\infty}^{\infty} \ln|S_o(j\omega)| \frac{1}{1+(\alpha-\omega)^2}\, d\omega < \ln|S_{max}| \underbrace{\int_{-\infty}^{-\omega_\ell} \frac{1}{1+(\alpha-\omega)^2}\, d\omega}_{I_1} + \\ + \ln(\epsilon) \underbrace{\int_{-\omega_\ell}^{\omega_\ell} \frac{1}{1+(\alpha-\omega)^2}\, d\omega}_{I_2} + \ln|S_{max}| \underbrace{\int_{\omega_\ell}^{\infty} \frac{1}{1+(\alpha-\omega)^2}\, d\omega}_{I_3} \tag{20}$$

We can now compute the integrals I_1, I_2 *and* I_3 *using MAPLE. This yields*

$$I_1 = \frac{\pi}{2} - \arctan(\alpha + \omega_\ell) \tag{21}$$

$$I_2 = \arctan(-\alpha + \omega_\ell) + \arctan(\alpha + \omega_\ell) \tag{22}$$

$$I_3 = \frac{\pi}{2} - \arctan(-\alpha + \omega_\ell) \tag{23}$$

Combining the above expressions we have that

$$\ln|S_{max}| > |\ln(\epsilon)| \frac{1}{f(\alpha) - 1} \tag{24}$$

where

$$f(\alpha) \triangleq \frac{\pi}{\arctan(-\alpha + \omega_\ell) + \arctan(\alpha + \omega_\ell)} > 1 \quad \textit{for all finite } \omega_\ell \textit{ and all finite } \alpha \tag{25}$$

In conclusion, the following observations are seen to apply:

- *The function* $f(\alpha)$ *is an even function of* α*. Thus the above result is also valid for the NMP zero* c_2*.*
- *The maximum lower bound occurs when* $\alpha = 0$*. This can be proved by differentiating* $f(\alpha)$ *with respect to* α
- *When* $\alpha >> \omega_\ell$*, we have that the lower bound reaches a minimum, since then* $f(\alpha)$ *goes to infinity.*

Solution 9.5. *We use the Poisson formula as in Lemma 9.5 to compute a lower bound for* S_{max}*.*
We first identify the following parameters and expressions:

$$\epsilon = 0.1; \qquad \omega_\ell = 1; \qquad \omega_h = 8 \tag{26}$$

$$B_p(s) = \frac{s-2}{s+2}; \qquad B_z(s) = \frac{s-2}{s+2} \tag{27}$$

Then, using equation (9.4.20) of the book

$$\ln S_{max} > \frac{1}{\Omega(4,8) - \Omega(4,1)} \left[|\pi \ln |B_p(4)|| + |(\ln 0.1)\Omega(4,1)| - (\pi - \Omega(4,8)) \ln(1.1) \right] \tag{28}$$

The lower limit for S_{max} can then be computed using the MATLAB routine ***smax*** *(provided on the CD-ROM in the book). This gives $S_{max} > 24.8473$.*
