Achieving the Multiple Description Rate-Distortion Region with Lattice Quantization

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Outline

- Practical Multiple Description Schemes.
- Successive Quantization and Quantization Splitting.
- Gaussian Multiple Description Rate-Distortion Region.
- Gram-Schmidt Orthogonalization.
- Simulate GS Orthogonalization by Sequential Quantization.
- Conclusion.

Practical Multiple Description Schemes

- MD Scalar/Vector Quantization: Vaishampayan *et al*.

  ◇ Index Assignment ◇ Lattice/Sublattice

- Entropy-Coded Dithered Lattice Quantizers (ECDQs): Frank-Dayan & Zamir 02.

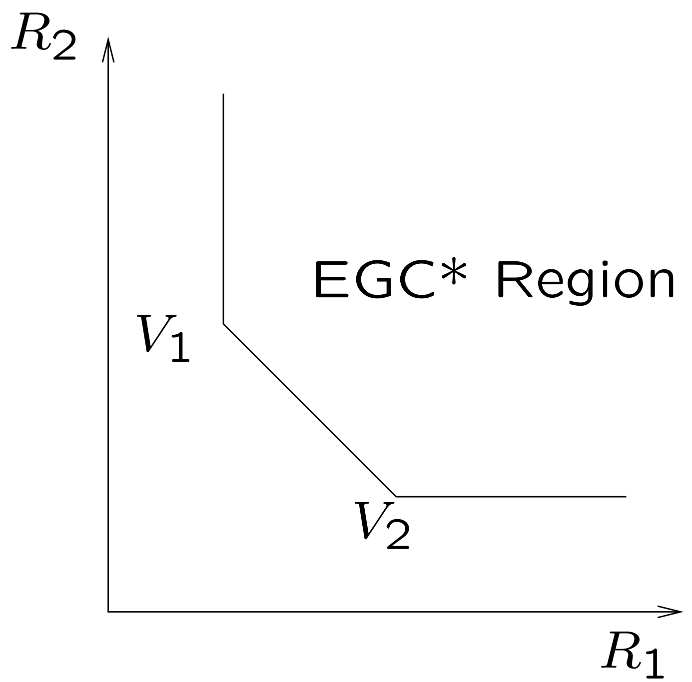
  ◇ Amenable to Analysis

- Correlating Transforms: Orchard *et al*. 97, Pradhan & Ramchandran 00, Goyal & Kovačević 01.

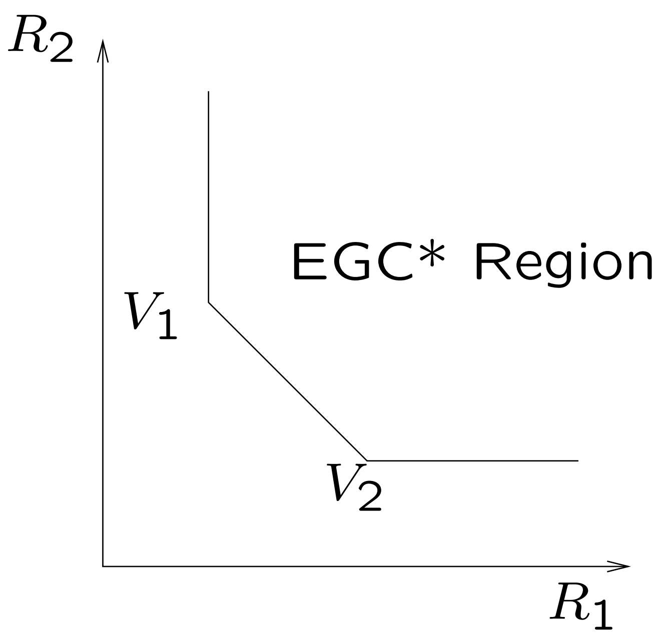
  ◇ Vector Sources

EGC* Region

$$
\begin{aligned}
R_i &\geq I(X;U_i), \quad i = 1,2, \\
R_1 + R_2 &\geq I(X;U_1) + I(X;U_2) + I(U_1;U_2|X).
\end{aligned}
$$



Successive Quantization



- Dominant Face: $R_1 + R_2 = I(X;U_1) + I(X;U_2) + I(U_1;U_2|X)$.

- V_1: $R_1 = I(X;U_1)$, $R_2 = I(X,U_1;U_2)$.
  V_2: $R_1 = I(X,U_2;U_1)$, $R_2 = I(X;U_2)$.

Quantization Splitting

- For any rate pair (R_1, R_2) on the dominant face, there exists a random variable U_2' with $(X, U_1) \to U_2 \to U_2'$ such that

$$\begin{aligned} R_1 &= I(X, U_2'; U_1), \\ R_2 &= I(X; U_2') + I(X, U_1; U_2 | U_2'). \end{aligned}$$

Quantization Order: $U_2' \to U_1 \to U_2$.

◇ If U_2' is independent of U_2, then

$$R_1 = I(X; U_1), \quad R_2 = I(X, U_1; U_2),$$

which are the coordinates of V_1.

◇ If $U_2' = U_2$, then

$$R_1 = I(X, U_2; U_1), \quad R_2 = I(X, U_2),$$

which are the coordinates of V_2.

Gaussian MD Rate-Distortion Region

Let

$$\begin{aligned} U_1 &= X + T_0 + T_1, \\ U_2 &= X + T_0 + T_2, \end{aligned}$$

where (T_1, T_2), T_0, X are zero-mean, jointly Gaussian and independent, and $\mathbb{E}(T_1 T_2) = -\sigma_{T_1}\sigma_{T_2}$. Let $\widehat{X}_i = \mathbb{E}(X|U_i) = \alpha_i U_i$ $(i = 1, 2)$, and $\widehat{X}_3 = \mathbb{E}(X|U_1, U_2) = \beta_1 U_1 + \beta_2 U_2$. Set $\mathbb{E}(X - \widehat{X}_i)^2 = D_i$, $i = 1, 2, 3$, then

$$\begin{aligned} \sigma_{T_0}^2 &= \frac{D_3 \sigma_X^2}{\sigma_X^2 - D_3}, \\ \sigma_{T_i}^2 &= \frac{D_i \sigma_X^2}{\sigma_X^2 - D_i} - \frac{D_3 \sigma_X^2}{\sigma_X^2 - D_3}, \quad i = 1, 2. \end{aligned}$$

With these $\sigma_{T_i}^2$ $(i = 0, 1, 2)$, it is straightforward to verify that

$$\begin{aligned} & I(X; U_i) = \frac{1}{2} \log \frac{\sigma_X^2}{D_i} \quad i = 1, 2, \\ & I(X; U_1) + I(X; U_2) + I(U_1; U_2|X) \\ = \; & \frac{1}{2} \log \frac{\sigma_X^2}{D_3} + \frac{1}{2} \log \frac{(\sigma_X^2 - D_3)^2}{(\sigma_X^2 - D_3)^2 - \left[\sqrt{(\sigma_X^2 - D_1)(\sigma_X^2 - D_2)} - \sqrt{(D_1 - D_3)(D_2 - D_3)}\right]^2}. \end{aligned}$$

The Correlation of Quantization Errors

We may view U_1 and U_2 as two different quantization of X.

$$\begin{aligned} U_1 &= X + T_0 + T_1, \\ U_2 &= X + T_0 + T_2, \end{aligned}$$

$$\begin{aligned} \mathbb{E}[(U_1 - X)(U_2 - X)] &= \mathbb{E}[(T_0 + T_1)(T_0 + T_2)] \\ &= \sigma_{T_0}^2 - \sigma_{T_1}\sigma_{T_2} \\ &= \frac{D_3\sigma_X^2}{\sigma_X^2 - D_3} - \sqrt{\left(\frac{D_1\sigma_X^2}{\sigma_X^2 - D_1} - \frac{D_3\sigma_X^2}{\sigma_X^2 - D_3}\right)\left(\frac{D_2\sigma_X^2}{\sigma_X^2 - D_2} - \frac{D_3\sigma_X^2}{\sigma_X^2 - D_3}\right)} \\ &\neq 0. \end{aligned}$$

It is hard to design two quantizers with quantization errors correlated in a desired manner.

Successive Quantization and GS Orthogonalization

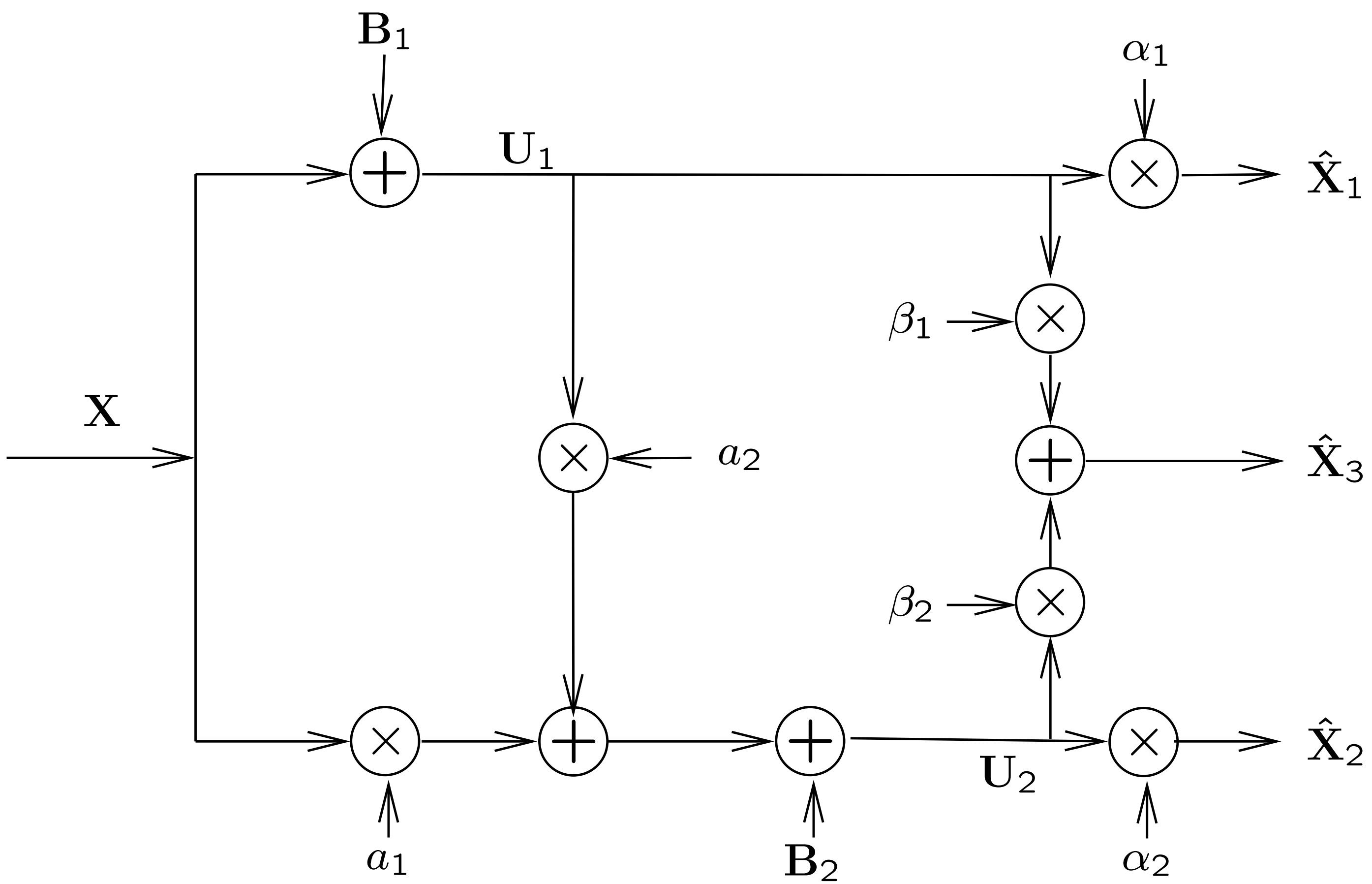
- GS Orthogonalization on (X, U_1, U_2).

$$
\begin{aligned}
B_0 &= X, \\
B_1 &= U_1 - \mathbb{E}(U_1|X) = U_1 - X, \\
B_2 &= U_2 - \mathbb{E}(U_1|X, U_1) = U_2 - a_1 X - a_2 U_1.
\end{aligned}
$$

- Successive Quantization for V_1.

$$
\begin{aligned}
R_1 &= I(X; U_1) = I(X; X + B_1), \\
R_2 &= I(X, U_1; U_2) = I(\mathbb{E}(U_2|X, U_1); U_2) \\
&= I(a_1 X + a_2 U_1; a_1 X + a_2 U_1 + B_2).
\end{aligned}
$$

Graph Representation of Successive Quantization



Quantization Splitting and GS Orthogonalization

- $U_2' = U_2 + T_3$.

- GS Orthogonalization on (X, U_2', U_1).

$$\begin{aligned}\widetilde{B}_0 &= X,\\ \widetilde{B}_1 &= U_1' - \mathbb{E}(U_2'|X) = U_2' - X,\\ \widetilde{B}_2 &= U_1 - \mathbb{E}(U_1|X, U_2') = U_1 - b_1X - b_2U_2'.\end{aligned}$$

- Quantization Splitting.

$$\begin{aligned}R_1 &= I(X, U_2'; U_1) = I(\mathbb{E}(U_1|X, U_2'); U_1)\\ &= I(b_1X + b_2U_2'; b_1X + b_2U_2' + \widetilde{B}_2),\\ R_2 &= I(X; U_2') + I(X, U_1; U_2|U_2')\\ &= I(X; X + \widetilde{B}_1) + I(X, U_1; U_2|U_2').\end{aligned}$$

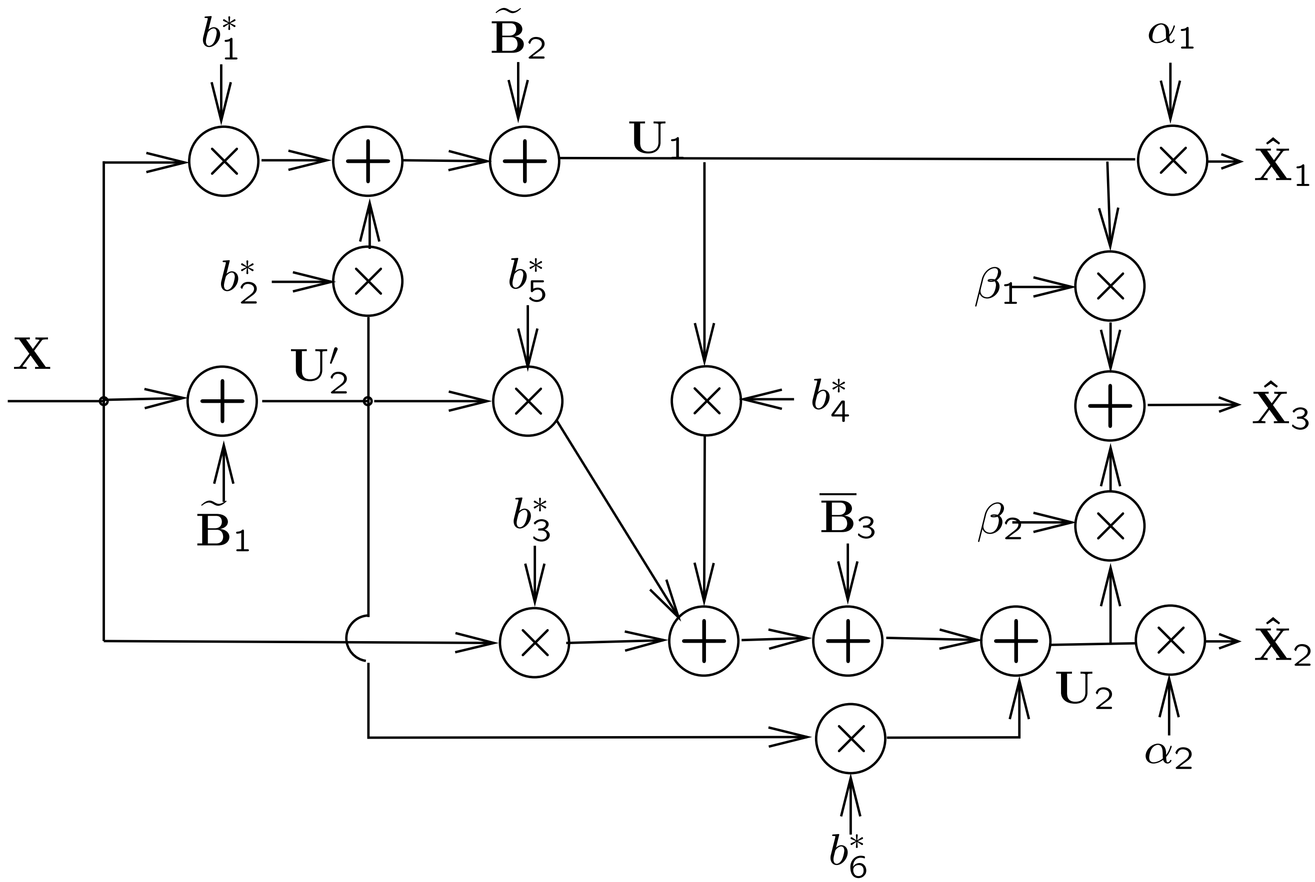
GS Orthogonalization on (U_2', X, U_1, U_2)

$$\begin{aligned}
\overline{B}_0 &= U_2', \\
\overline{B}_1 &= X - \mathbb{E}(X|\overline{B}_0) = X - b_3\overline{B}_0, \\
\overline{B}_2 &= U_1 - \mathbb{E}(U_1|\overline{B}_0) - \mathbb{E}(U_1|\overline{B}_1) = U_1 - b_4\overline{B}_0 - b_5\overline{B}_1, \\
\overline{B}_3 &= U_2 - \mathbb{E}(U_2|\overline{B}_0) - \mathbb{E}(U_2|\overline{B}_1) - \mathbb{E}(U_2|\overline{B}_2) \\
&= U_2 - b_6\overline{B}_0 - b_7\overline{B}_1 - b_8\overline{B}_2.
\end{aligned}$$

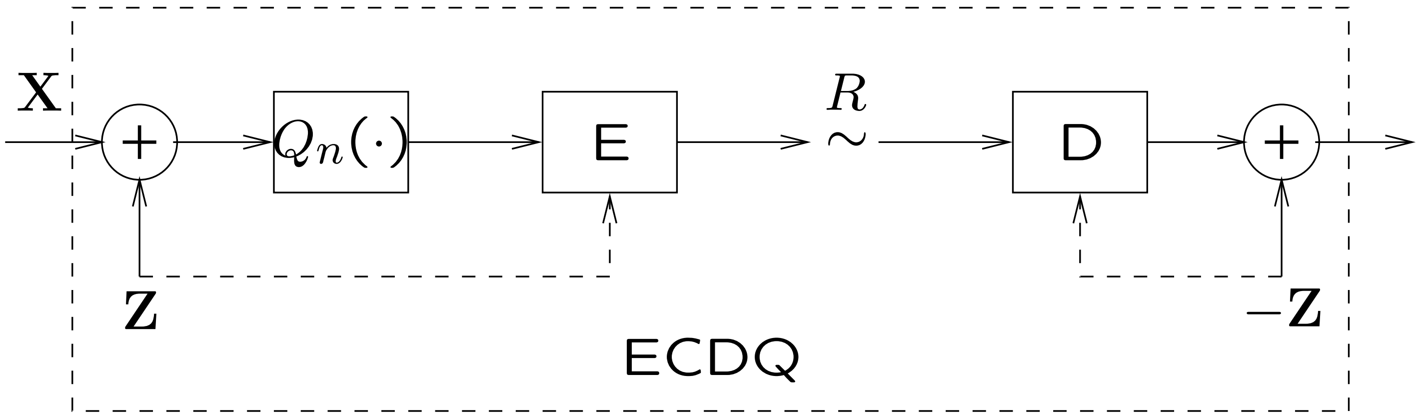
$$\begin{aligned}
& I(X, U_1; U_2|U_2') \\
= \; & I(b_3\overline{B}_0 + \overline{B}_1, b_4\overline{B}_0 + b_5\overline{B}_1 + \overline{B}_2; b_6\overline{B}_0 + b_7\overline{B}_1 + b_8\overline{B}_2 + \overline{B}_3|\overline{B}_0) \\
= \; & I(\overline{B}_1, b_5\overline{B}_1 + \overline{B}_2; b_7\overline{B}_1 + b_8\overline{B}_2 + \overline{B}_3) \\
= \; & I(b_7\overline{B}_1 + b_8\overline{B}_2; b_7\overline{B}_1 + b_8\overline{B}_2 + \overline{B}_3).
\end{aligned}$$

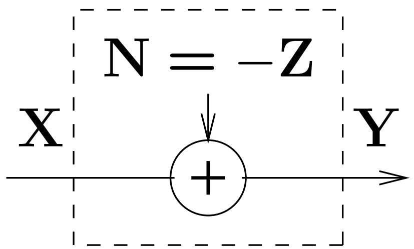
Note: $b_7\overline{B}_1 + b_8\overline{B}_2 = (b_7 - b_5 b_8)X + b_8 U_1 + (b_3 b_5 b_8 - b_3 b_7 - b_4 b_8)U_2'$.

Graph Representation of Quantization Splitting



Entropy-Coded Dithered Lattice Quantization





- $Q_n(\mathbf{X}+\mathbf{Z})-\mathbf{Z} \quad \sim \quad \mathbf{X}+\mathrm{N}.$
- $R = H(Q_n(\mathbf{X}+\mathbf{Z})|\mathbf{Z}) = I(\mathbf{X};\mathbf{Y}) = h(\mathbf{Y}) - h(\mathbf{N}).$

Simulate GS Orthogonalization by Sequential (Dithered) Quantization

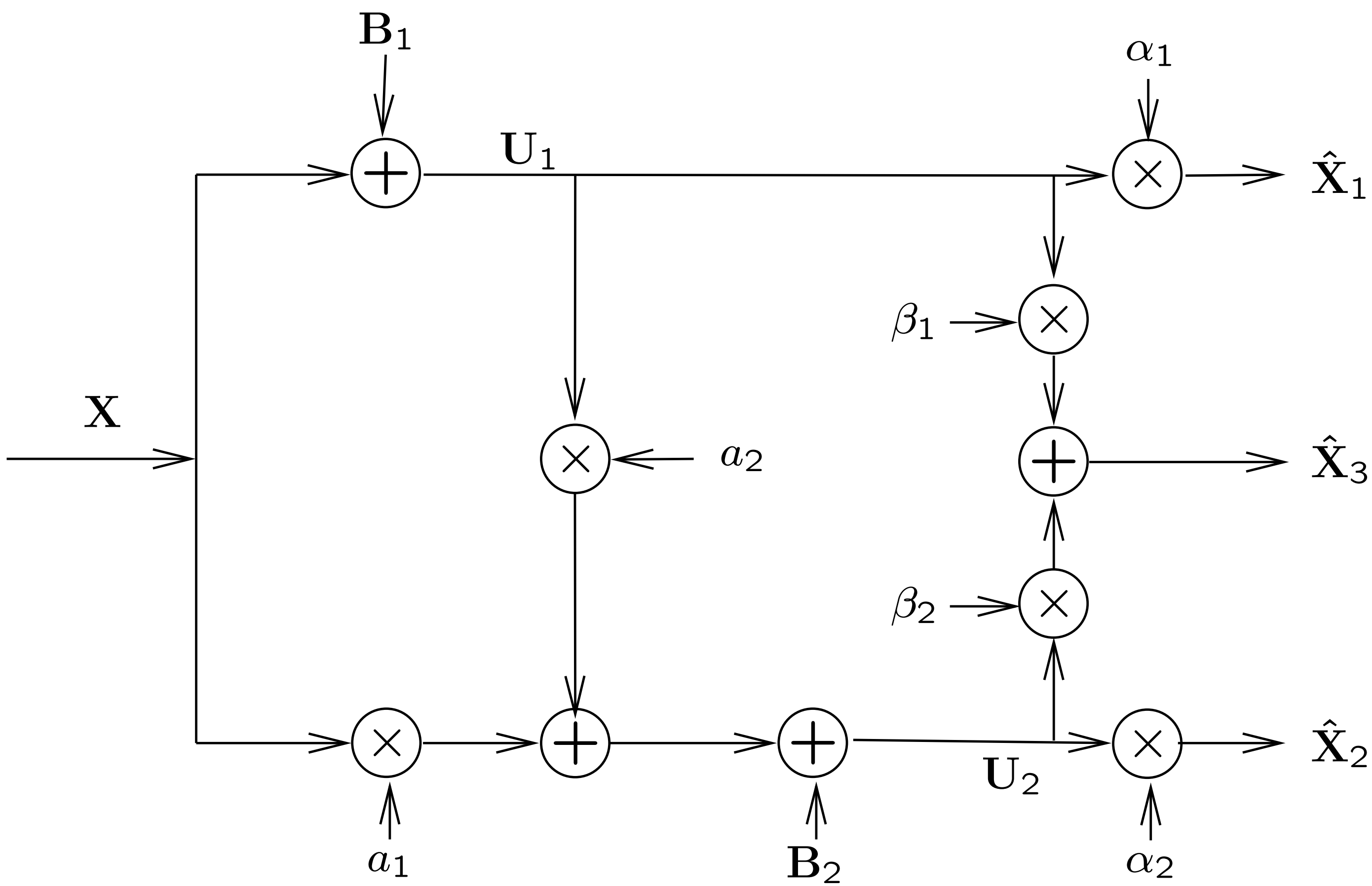
$$\begin{aligned} I_0 &= X_1, \\ I_1 &= X_2 - a_1 X_1, \\ I_2 &= X_3 - a_2 X_1 - a_3 X_2, \\ &\vdots \end{aligned}$$

Let Z_i have the same variance as I_i, $i = 1, 2, \cdots$. Construct the following sequential (dithered) quantization system.

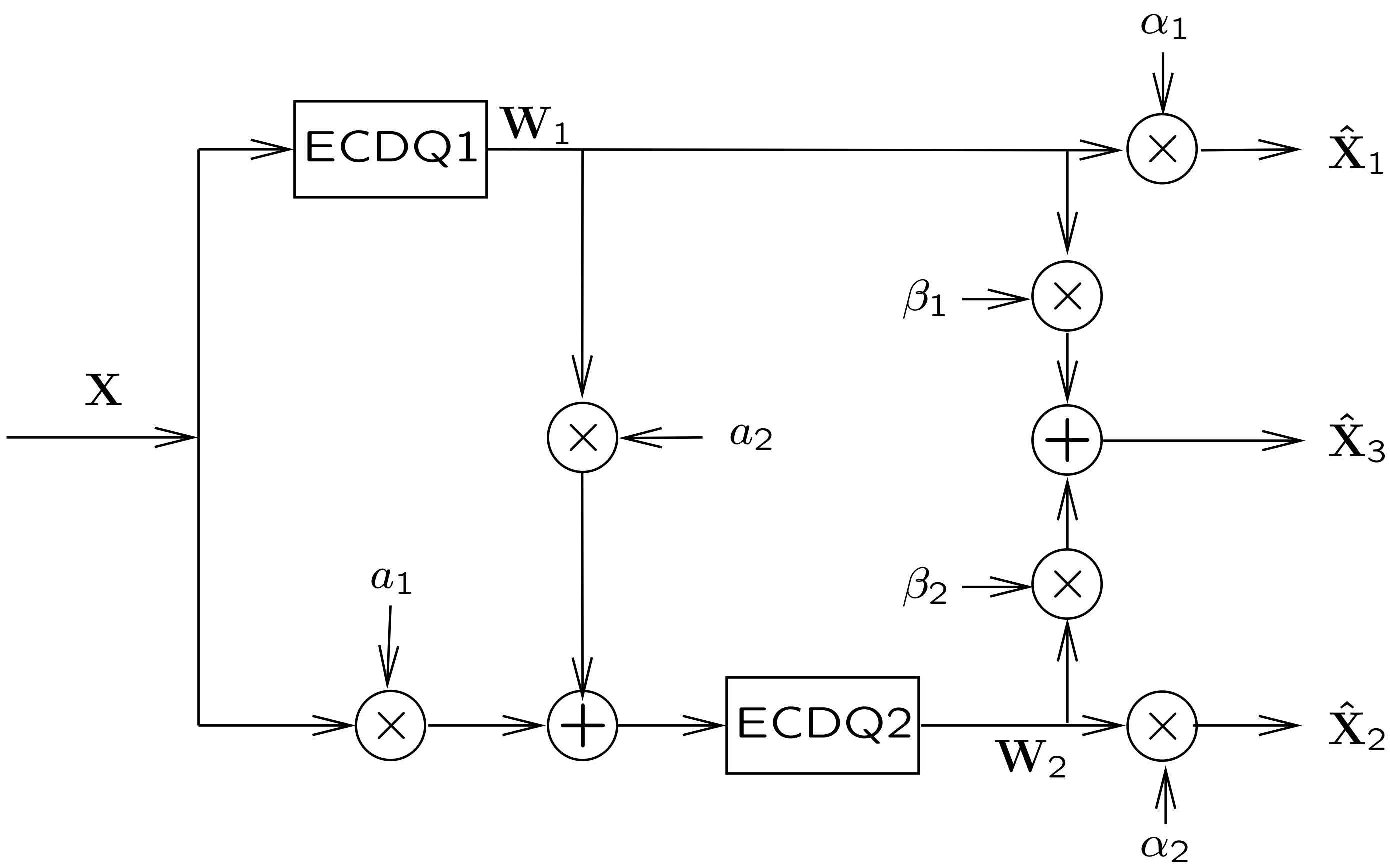
$$\begin{aligned} \widetilde{X}_1 &= X_1, \\ \widetilde{X}_2 &= Q_1(a_1 \widetilde{X}_1 + Z_1) - Z_1, \\ \widetilde{X}_3 &= Q_2(a_2 \widetilde{X}_1 + a_3 \widetilde{X}_2 + Z_2) - Z_2, \\ &\vdots \end{aligned}$$

$(X_1, X_2, \cdots,)$ and $(\widetilde{X}_1, \widetilde{X}_2, \cdots)$ have the same covariance matrix.

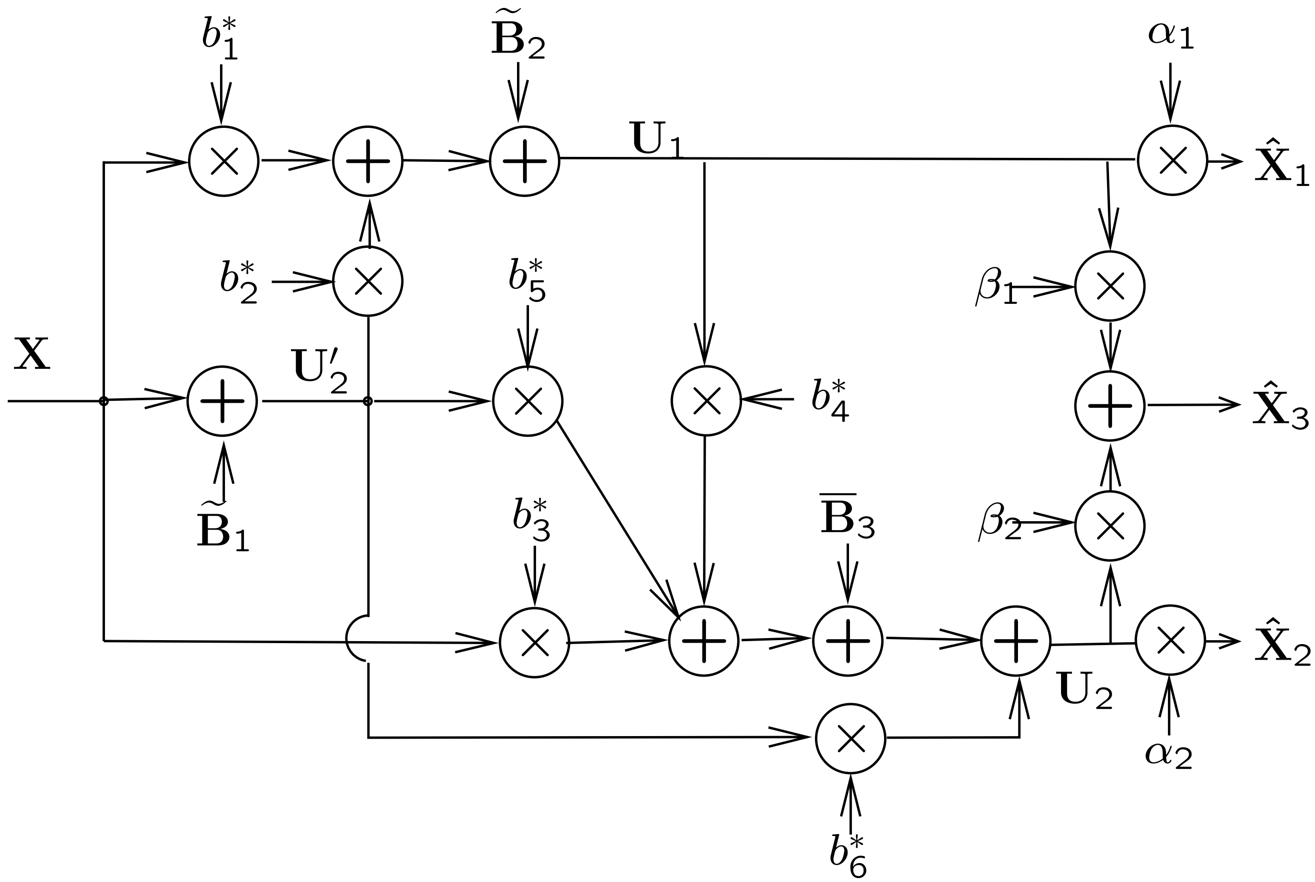
Successive Quantization — Additive Noise



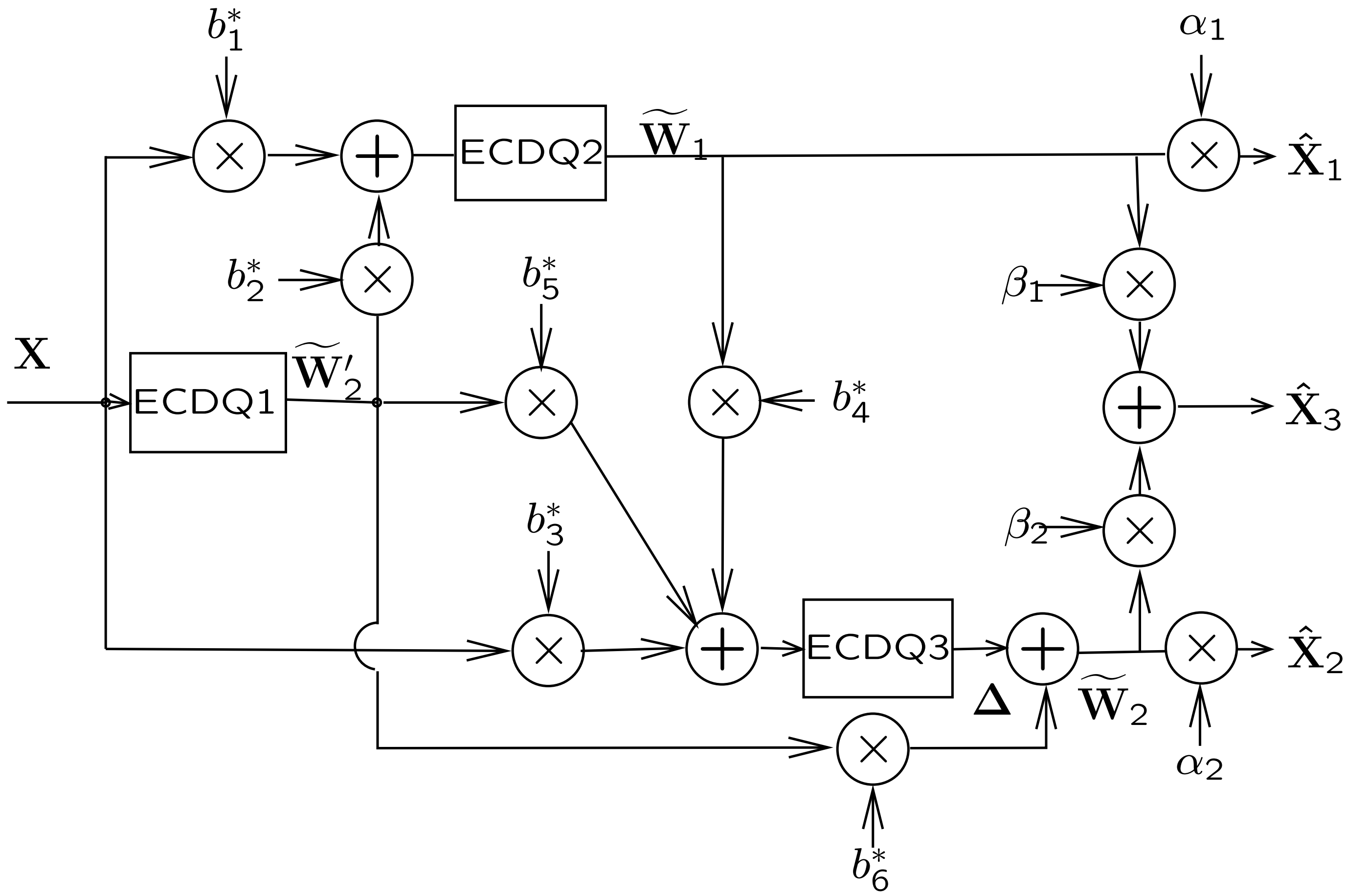
Successive Quantization — ECDQ



Quantization Splitting — Additive Noise



Quantization Splitting — ECDQ



Property and Performance

- Achieve the whole Gaussian MD rate-distortion region as the dimension of the (optimal) lattice quantizers becomes large.

- For general smooth sources, the performance no worse than that for an i.i.d. Gaussian source with the same variance.

- Asymptotically optimal at high resolution for general smooth sources.

- Universal in the sense that it only needs the information of the first and second order statistics of the source.

- For the scalar case, the central and side distortion product is 2.596 dB away from the information theoretic distortion product; 2.5 dB if timesharing of vertices is used.

Conclusion

- A Framework for Practical MD Quantization Systems
  - ◇ Successive Quantization (Splitting) $\rightarrow$ Ordering
  - ◇ GS Orthogonalization $\rightarrow$ Sufficient Statistics
  - ◇ ECDQ $\rightarrow$ Practical Implementation

- Generalization to the n-Channel Case (Venkataramani *et al.* 03, Pradhan *et al.* 04).

  ◇ Successive Quantization (Splitting) ◇ Contra-polymatroid ◇ Duality

- MMSE Estimation and Lattice Coding/Quantization (Erez & Zamir 04). Shannon meets Wiener (Forney 03,04).