Q1: Use time-shifting theorem to determine $\mathcal{L}[f(t)]$.
where $f(t) = [t-1 + e^{-(t-1)}]\,u(t-1)$.

Sol. Let $g(t) = (t + e^{-t})\,u(t)$

$$G(s) = \frac{1}{s^2} + \frac{1}{s+1}$$

Then $F(s) = e^{-s} G(s)$

$$F(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s+1}\right)$$

Q2: Given $F(s)$, find $f(t)$

(a) $F(s) = \dfrac{4}{(s+3)(s+4)}$

Sol:

① $F(s) = \dfrac{A}{s+3} + \dfrac{B}{s+4}$

② $A = F(s)(s+3)\Big|_{s=-3} = \dfrac{4}{-3+4} = 4$

$B = F(s)(s+4)\Big|_{s=-4} = \dfrac{4}{-4+3} = -4$

③ $F(s) = \dfrac{4}{s+3} - \dfrac{4}{s+4}$

④ $f(t) = (4e^{-3t} - 4e^{-4t})\, u(t)$

(b) $F(s) = \dfrac{10s}{(s+1)(s+6)}$

Sol:

① $F(s) = \dfrac{A}{s+1} + \dfrac{B}{s+6}$

② $A = F(s)(s+1)\Big|_{s=-1} = \dfrac{10\cdot(-1)}{-1+6} = -2$

$B = F(s)(s+6)\Big|_{s=-6} = \dfrac{10\cdot(-6)}{-6+1} = 12$

③ $F(s) = \dfrac{-2}{s+1} + \dfrac{12}{s+6}$

④ $f(t) = (-2e^{-t} + 12e^{-6t})\, u(t)$

Q3: Given $F(s)$, find $f(t)$.

(a) $$F(s) = \frac{s^2+7s+12}{(s+2)(s+4)(s+6)}$$

Sol: $$F(s) = \frac{(s+3)(s+4)}{(s+2)(s+4)(s+6)} = \frac{s+3}{(s+2)(s+6)}$$

① $$F(s) = \frac{A}{s+2} + \frac{B}{s+6}$$

② $$A = F(s)(s+2)\Big|_{s=-2} = \frac{-2+3}{-2+6} = \frac{1}{4}$$

$$B = F(s)(s+6)\Big|_{s=-6} = \frac{-6+3}{-6+2} = \frac{3}{4}$$

③ $$F(s) = \frac{1}{4}\frac{1}{s+2} + \frac{3}{4}\frac{1}{s+6}$$

④ $$f(t) = \left(\frac{1}{4}e^{-2t} + \frac{3}{4}e^{-6t}\right)u(t)$$

(b) $$F(s) = \frac{s+1}{s(s+2)(s+3)}$$

Sol: ① $$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

② $$A = F(s)s\Big|_{s=0} = \frac{1}{6}$$

② $$B = F(s)(s+2)\Big|_{s=-2} = \frac{-2+1}{-2\cdot(-2+3)} = \frac{1}{2}$$

② $$C = F(s)(s+3)\Big|_{s=-3} = \frac{-3+1}{-3\cdot(-3+2)} = -\frac{2}{3}$$

③ $$F(s) = \frac{1}{6}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2} - \frac{2}{3}\frac{1}{s+3}$$

④ $$f(t) = \left(\frac{1}{6} + \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t}\right)u(t)$$

Q4: Given $F(s)$, find $f(t)$

(a) $F(s) = \dfrac{10}{s^2+2s+2}$

sol 1:

$$\frac{|A|\angle\theta}{s+\alpha-j\beta} + \frac{|A|\angle-\theta}{s+\alpha+j\beta} \xleftrightarrow{\mathcal{L}} 2|A|e^{-\alpha t}\cos(\beta t+\theta)\,u(t).$$

① $F(s) = \dfrac{A}{s+1-j} + \dfrac{A^*}{s+1+j}$

② $A = F(s)(s+1-j)\big|_{s=-1+j} = \dfrac{10}{-1+j+1+j} = -5j = 5\angle-90^\circ$

③ $|A| = 5,\quad \theta = -90^\circ,\quad \alpha = 1,\quad \beta = -1$

④ $f(t) = 10e^{-t}\cos(t-90^\circ)\,u(t)$

sol 2:

$$\sin\omega t \xleftrightarrow{\mathcal{L}} \frac{\omega}{s^2+\omega^2}, \quad e^{-at}f(t) \xleftrightarrow{\mathcal{L}} F(s+a).$$

$$e^{-at}\sin\omega t \xleftrightarrow{\mathcal{L}} \frac{\omega}{(s+a)^2+\omega^2}$$

① $F(s) = \dfrac{10}{(s+1)^2+1}$

② $f(t) = 10e^{-t}\sin t\,u(t).$

3-4 (b) $F(s) = \dfrac{10(s+2)}{s^2+4s+5}$

sol1: ① $F(s) = \dfrac{A}{s+2-j} + \dfrac{A^*}{s+2+j}$

② $A = F(s)(s+2-j)\Big|_{s=-2+j} = \dfrac{10 \cdot j}{-2+j+2+j} = j$

③ $|A| = 5, \quad \theta = 0°, \quad \alpha = 2, \quad \beta = 1$

④ $f(t) = 10 e^{-2t} \cos t \; u(t)$

Sol2: $\cos \omega t \overset{L}{\longleftrightarrow} \dfrac{s}{s^2+\omega^2}, \quad e^{-at} f(t) \overset{L}{\longleftrightarrow} F(s+a)$

$\Downarrow$

$e^{-at} \cos \omega t \overset{L}{\longleftrightarrow} \dfrac{s+a}{(s+a)^2+\omega^2}$

① $F(s) = \dfrac{10(s+2)}{(s+2)^2+1}$

② $f(t) = 10 e^{-2t} \cos t \; u(t)$

Ex: Given $F(s)$, find $f(t)$

(a) $F(s) = \dfrac{10(s+1)}{s^2+2s+2}$

Sol1:

① $F(s) = \dfrac{A}{s+1-j} + \dfrac{A^*}{s+1+j}$

② $A = F(s)(s+1-j)\big|_{s=-1+j} = \dfrac{10j}{2j} = 5$

③ $|A| = 5$, $\theta = 0°$, $\alpha = 1$, $\beta = 1$

④ $f(t) = 5e^{-t}\cos(t)\,u(t)$

Sol2:

① $F(s) = \dfrac{10(s+1)}{(s+1)^2+1}$

② $f(t) = 10e^{-t}\cos t\,u(t)$.

(b) $F(s) = \dfrac{s+1}{s(s^2+4s+5)}$

Sol1: ① $F(s) = \dfrac{A}{s} + \dfrac{B}{s+2-j} + \dfrac{B^*}{s+2+j}$

② $A = F(s)\,s\big|_{s=0} = \dfrac{1}{5} = 0.2$

$B = F(s)(s+2-j)\big|_{s=-2+j} = \dfrac{-1+j}{(-2+j)(2j)} = 0.632\angle -108.43°$

③ $f(t) = [0.2 + 0.632e^{-2t}\cos(t - 108.43°)]u(t)$

Q5. (b) Sol2:

① $F(s) = \dfrac{A}{s} + \dfrac{Bs + C}{s^2+4s+5}$

② $s+1 = A(s^2+4s+5) + (Bs+C)s$

compare coefficients:

$$\begin{cases} A+B=0 \\ 4A+C=1 \\ 5A=1 \end{cases} \Rightarrow \begin{array}{l} A=0.2 \\ B=-0.2 \\ C=0.2 \end{array}$$

③ $F(s) = \dfrac{0.2}{s} - \dfrac{0.2s-0.2}{s^2+4s+5}$

$= \dfrac{0.2}{s} - \dfrac{0.2(s+2)}{(s+2)^2+1} + \dfrac{0.6}{(s+2)^2+1}$

④ $f(t) = (0.2 - 0.2e^{-2t}\cos t + 0.6e^{-2t}\sin t)\,u(t)$

* Another way to find A, B, C

$s+1 = A(s^2+4s+5) + (Bs+c)s$

$$\begin{cases} \text{Let } s=0, & 1 = A\cdot 5 + 0 \\ \text{Let } s=1, & 2 = A\cdot 10 + (B+C) \\ \text{Let } s=-1, & 0 = A\cdot 2 + (-B+C)\cdot(-1) \end{cases} \Rightarrow \begin{cases} A=0.2 \\ B=-0.2 \\ C=0.2 \end{cases}$$

Q7: Given $F(s)$, find $f(t)$:

(a) $$F(s) = \frac{s+6}{s^2(s+2)}$$

Sol: ① $$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

② $$K_{rj} = \frac{1}{(r-j)!} \frac{d^{r-j}}{ds^{r-j}} \left[F(s)(s+p)^r \right] \Big|_{s=-p}$$

$$B = F(s) \cdot s^2 \Big|_{s=0} = \frac{6}{2} = 3$$

$$A = \frac{1}{1!} \frac{d}{ds} \left(F(s) s^2 \right) \Big|_{s=0} = -1$$

$$C = F(s)(s+2) \Big|_{s=-2} = \frac{4}{4} = 1$$

③ $$F(s) = -\frac{1}{s} + \frac{3}{s^2} + \frac{1}{s+2}$$

$$\frac{1}{s^{n+1}} \xleftrightarrow{L} \frac{t^n}{n!}$$

$$f(t) = (-1 + 3t + e^{-2t})\, u(t).$$

(b) $$F(s) = \frac{s+3}{(s+1)^2(s+3)}$$

Sol: $$F(s) = \frac{1}{(s+1)^2}$$

$$f(t) = te^{-t} u(t).$$

Q8: Given $F(s)$, find $f(t)$.

$$F(s) = \frac{(s^2+2s+1)e^{-2s}}{s(s+1)(s+2)}$$

Sol:

$$F(s) = \frac{(s+1)^2 e^{-2s}}{s(s+2)\cancel{(s+1)}} = \frac{(s+1)e^{-2s}}{s(s+2)}$$

Let $G(s) = \frac{s+1}{s(s+2)}$, $\quad F(s) = G(s)\cdot e^{-2s}$

then $f(t) = g(t-2)$ $\qquad e^{-as}F(s) \xleftrightarrow{L} f(t-a)$.

① $G(s) = \frac{A}{s} + \frac{B}{s+2}$

② $A = G(s)\, s \,|_{s=0} = 0.5$

$B = G(s)(s+2)\,|_{s=-2} = 0.5$

③ $G(s) = \frac{0.5}{s} + \frac{0.5}{s+2}$

④ $g(t) = (0.5 + 0.5e^{-2t})\, u(t)$

⑤ $f(t) = (0.5 + 0.5e^{-2(t-2)})\, u(t-2)$.

Q9: Solve differential equations using Laplace transforms.

(a) $\frac{d^2y(t)}{dt} + \frac{4dy(t)}{dt} + 4y(t) = u(t)$, $y(0)=0$, $y'(0)=1$

Sol: $\frac{d^n f(t)}{dt^n} \xleftrightarrow{L} s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

① $s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 4Y(s) = \frac{1}{s}$

② $Y(s)(s^2+4s+4) = 1+\frac{1}{s}$

③ $Y(s) = \frac{s+1}{s(s+2)^2}$

④ $Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$

⑤ $s+1 = A(s+2)^2 + Bs(s+2) + Cs$

$$\begin{cases} \text{Let } s=0, & 1 = 4A+0+0 \\ \text{Let } s=1, & 2 = 9A+3B+C \\ \text{Let } s=-2, & -1 = 0+0+(-2C) \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \\ C = \frac{1}{2} \end{cases}$$

⑥ $Y(s) = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}}{s+2} + \frac{\frac{1}{2}}{(s+2)^2}$

⑦ $y(t) = (\frac{1}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t})u(t)$

Q9: (b) $\frac{dy(t)}{dt} + 3y(t) + 2\int_0^t y(x)\,dx = u(t)$, $y(0)=0$, $t>0$.

sol: $\int_0^t f(\lambda)\,d\lambda \xleftrightarrow{L} \frac{1}{s}F(s)$.

① $sY(s) - y(0) + 3Y(s) + \frac{2}{s}Y(s) = \frac{1}{s}$

② $Y(s)\left(s+3+\frac{2}{s}\right) = \frac{1}{s}$

$$Y(s) = \frac{1}{(s+2)(s+1)}$$

③ $Y(s) = \frac{A}{s+2} + \frac{B}{s+1}$

$A = Y(s)(s+2)\big|_{s=-2} = -1$

$B = Y(s)(s+1)\big|_{s=-1} = 1$

④ $Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$

⑤ $y(t) = (e^{-t} - e^{-2t})\,u(t)$.

6.10: Given $F(s)$, determine the initial and final values of $f(t)$.

(a) $F(s) = \frac{2(s+2)}{s(s+1)}$

Sol: $f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{2(s+2)}{s+1} = 2$

$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2(s+2)}{s+1} = 4$

(b) $F(s) = \frac{2(s^2+2s+6)}{(s+1)(s+2)(s+3)}$

Sol: $f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{2s(s^2+2s+6)}{(s+1)(s+2)(s+3)} = 2$

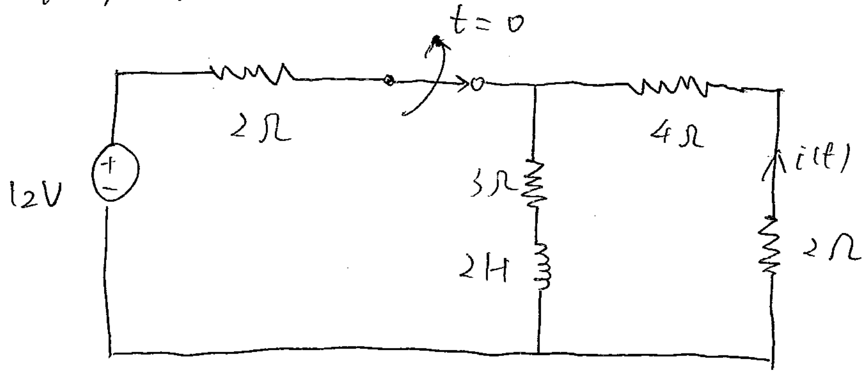
$f(\infty) = \lim_{s\to 0} sF(s) = 0$

(c) $F(s) = \frac{2s^2}{(s+1)(s^2+2s+2)}$

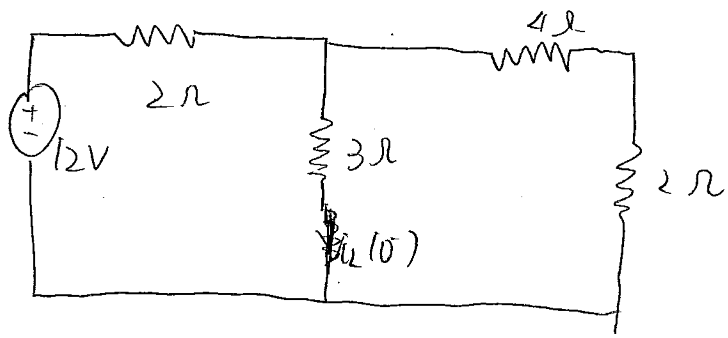
Sol: $f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{2s^3}{(s+1)(s^2+2s+2)} = 2$

$f(\infty) = \lim_{s\to 0} sF(s) = 0$

Q11: Find $i(t)$, $t>0$



Sol: when $t<0$,

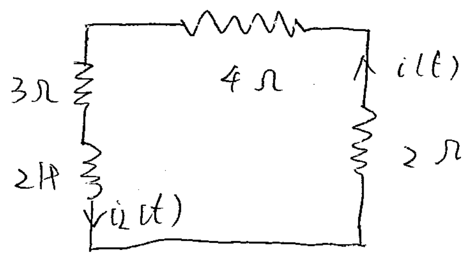


$$R_{eq} = [(4+2) // 3] + 2 = 4\ \Omega$$

$$I = \frac{12\text{ V}}{R_{eq}} = 3A$$

$$i_L(0^-) = \frac{4+2}{4+2+3} \cdot 3 = 2A$$

When $t>0$



inductor:

$$v(t) = L\frac{di(t)}{dt}$$

① $$2\frac{di(t)}{dt} + 2i(t) + 4i(t) + 3i(t) = 0$$

② $$2[sI(s) - i(0^-)] + 2I(s) + 4I(s) + 3I(s) = 0$$

$$I(s)\,(2s+2+4+3) = 4$$

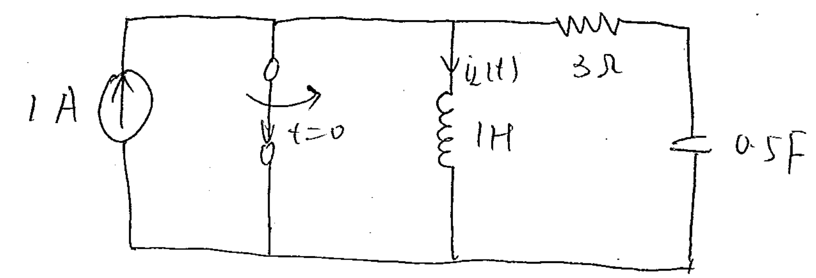
011.

$$I(s) = \frac{4}{2s+9}$$

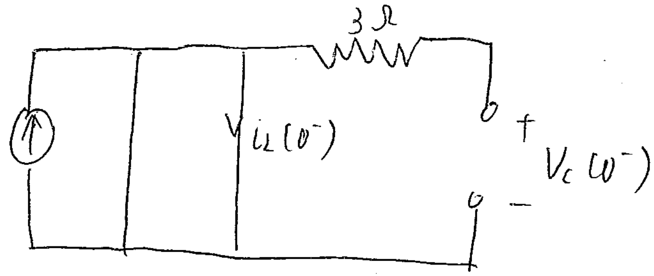
(3) $I(s) = \dfrac{2}{s+\frac{9}{2}}$.

(4) $i(t) = 2e^{-\frac{9}{2}t}\,u(t)$ A, $t>0$,

Q12: find $i_L(t)$, for $t>0$,

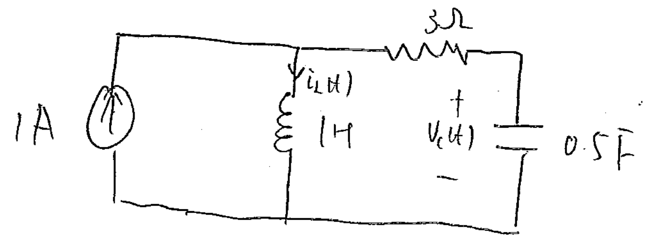


sol. when $t<0$



$i_L(0^-)=0$ A, $\quad V_c(0^-)=0$ V

When $t>0$



capacitor:

$i_c(t) = C\dfrac{dv_c(t)}{dt}$

① $\begin{cases} i_L(t)+0.5\dfrac{dV_c(t)}{dt} = 1 \\ \dfrac{di_L(t)}{dt} = V_c(t)+3\cdot 0.5\cdot\dfrac{dV_c(t)}{dt} \end{cases}$

② $\begin{cases} I_L(s) + 0.5[s\,V_c(s) - V_c(0^-)] = \dfrac{1}{s} \\ s\,I_L(s) - i_L(0^-) = V_c(s) + 1.5[s\,V_c(s) - V_c(0^-)] \end{cases}$

③ $I_L(s) = \dfrac{3s+2}{s(s+1)(s+2)}$

Q12:

④ $I_L(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

⑤ $A = I_L(s)\, s\big|_{s=0} = 1$

$B = I_L(s)\,(s+1)\big|_{s=-1} = \frac{-3+2}{(-1)(-1+2)} = 1$

$C = I_L(s)\,(s+2)\big|_{s=-2} = \frac{-6+2}{-2\cdot(-2+1)} = -2$

⑥ $I_L(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{2}{s+2}$

⑦ $i_L(t) = (1 + e^{-t} - 2e^{-2t})\, u(t)$ A