Laboratory 5: First-Order Circuits and Linear Differential Equations

ELEC ENG 2CJ4: Circuits and Systems
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1 Objective

First-order circuits are fundamental building blocks of many important devices. In this lab, you will learn how to analyze such circuits.

2 Euqipment

The following equipments are used in this laboratory:

- DC voltage source with positive and negative output($\pm$9V); Oscilloscope; Function signal generator
- Op-Amp LM358
- Resistors: $10k\Omega \times 2$, $25k\Omega \times 1$, $100k\Omega \times 1$
- Capacitors: $10nF(103) \times 1$, $100nF(104) \times 1$
- Diode: Standard $1N4148 \times 2$

3 Introduction to First-Order Linear Differential Equations

3.1 Natural and Forced Responses

Consider the following first-order differential equation with constant coefficients a_0 and a_1 such that $a_1 \neq 0$,

$$a_1 x'(t) + a_0 x(t) = f(t). \tag{1}$$

The solution to (1) is denoted by $x(t) = x_n(t) + x_f(t)$, where $x_n(t)$ and $x_f(t)$ are called the natural response and the forced response, respectively. In particular, $x_n(t)$ is the solution to the following homogeneous equation

$$a_1 x'(t) + a_0 x(t) = 0, \tag{2}$$

and it takes the form of $x_n(t) = Ke^{St}$, where K and S are constants. Substituting the expression of $x_n(t)$ back to (2), we have

$$(a_1 S + a_0)Ke^{St} = 0. \tag{3}$$

As $K \neq 0$ and $e^{St} > 0$ (otherwise the solution is trivial), we have

$$S = -\frac{a_0}{a_1}. \tag{4}$$

The value of K can be determined by the initial condition on $x(0^+)$ and $f(0^+)$, which will be discussed later. Some functional forms of the forced solution are listed here for your reference.

Table 1: Forms of forced solutions

$f(t)$	Trial Function
a	A
$at + b$	$At + B$
$at^n + bt^{n-1} + \dots$	$At^n + Bt^{n-1} + \dots$
$ae^{\sigma t}$	$Ae^{\sigma t}$
$a\cos\omega t + b\sin\omega t$	$A\cos\omega t + B\sin\omega t$

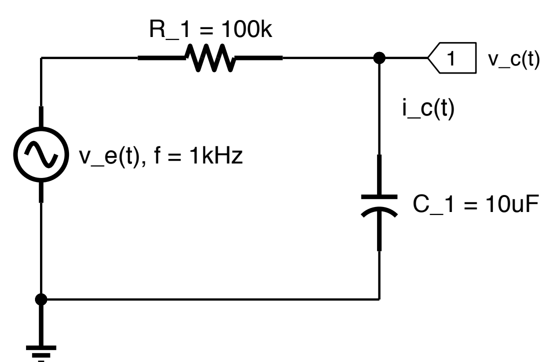

Figure 1: A first-order RC circuit

3.2 Solving the First-Order RC Circuits with DC Source

Consider the first-order RC circuit depicted in Fig. 1. It follows by the KVL and the property of capacitor that

$$R_1 i_c(t) + v_c(t) = v_e(t), \tag{5a}$$
$$i_c(t) = C_1 v_c'(t), \tag{5b}$$

where $i_c(t)$ is the charging current of C_1, $v_c(t)$ is the end-to-end voltage of C, and $v_e(t)$ is the value of the voltage source. Hence the voltage $v_c(t)$ can be fully characterized by the first order differential equation

$$R_1 C_1 v_c'(t) + v_c(t) = v_e(t). \tag{6}$$

If we know the expression of $v_e(t)$ and the initial conditions, then $v_c(t)$ can be solved in closed-form.

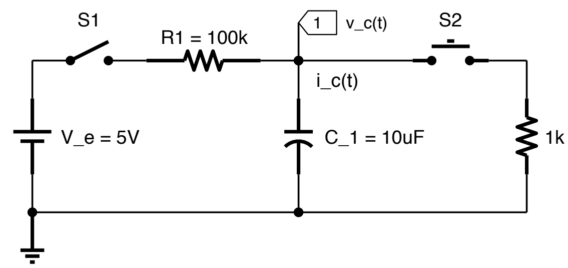

Figure 2: A first-order RC circuit with DC source

In what follows, we consider the first-order circuit with DC source (step response) in Fig. 2. First press S_2 for a while, and then close switch S_1 at time $t = t_0$. We assume that

$$v_c(t = t_0^+) = V_c,$$
$$v_e(t = t_0^+) = V_e,$$

where V_c is the initial voltage of C_1. Clearly, if we press S_2 for sufficiently long time with S_1 open-loop, then $V_c = 0$. From (6), for $t \geq t_0^+$, we have

$$R_1 C_1 v_c'(t) + v_c(t) = v_e(t), \quad t \geq t_0^+, \tag{7}$$

where

$$v_e(t) = \begin{cases} 0, & t < t_0, \\ V_e, & t \geq t_0. \end{cases}$$

In view of (4), the natural response to $v_c(t)$ is given by

$$v_{c,n}(t) = Ke^{S(t-t_0)}, \tag{8}$$

where $S = -\frac{1}{R_1 C_1}$, and K can be determined by the initial conditions.
From Table 1, we find that the forced response is a constant number; as such we denote $v_{c,f}(t) = A$, $t > t_0^+$. Since

$$R_1 C_1 v_{c,f}'(t) + v_{c,f}(t) = V_e, \quad t \geq t_0^+,$$

it follows that $v_{c,f}(t) = V_e$, $t \geq t_0^+$. Note that

$$v_c(t) = v_{c,n}(t) + v_{c,f}(t) = Ke^{-\frac{t-t_0}{R_1C_1}} + V_e. \tag{9}$$

For $t = t_0^+$, we have $v_c(t_0^+) = K + V_e = V_c$, which implies $K = V_c - V_e$. Therefore,

$$v_c(t) = (V_c - V_e)e^{-\frac{t-t_0}{R_1C_1}} + V_e, \quad t > t_0^+, \tag{10}$$

which is illustrated in Fig. 3.

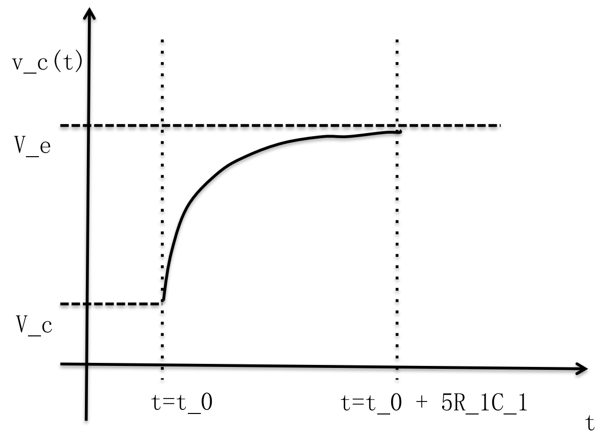

Figure 3: Charging process for the first-order RC circuit with DC source

3.3 First-order RC Circuit in the Relaxation Oscillator

The relaxation oscillator is an important circuit that uses a Schmitt trigger to alternately charge and discharge the capacitor. And it is widely used in low frequency function signal generators. In this experiment, we will study the first-order RC circuit in the relaxation oscillator.

3.3.1 The Principle of the Relaxation Oscillator

Consider the relaxation oscillator depicted in Fig. 4, where the Schmitt trigger is used to alternately charge and discharge the capacitor C through the resistor R_3.

Without loss of generality, we suppose that, initially, the output $v_{out}(t) = V_{sat+}$; as a consequence, $v_c(t) < V_{th+} < V_{sat+}$ (Can you explain why?), and the capacitor will be charged. When $v_c(t)$ reaches V_{th+}, the output voltage reverses to its opposite saturation limit V_{sat-} and the trigger threshold changes to V_{th-}. As the output is negative now and $v_c(t) > V_{th-} > V_{sat-}$, the capacitor will be discharged by R_3. Similarly, when $v_c(t)$ decreases to V_{th-}, the output would

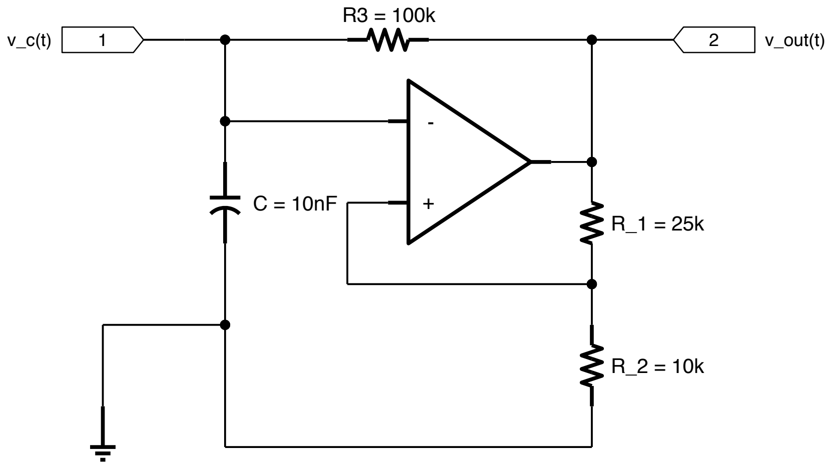

Figure 4: A simple relaxation oscillator

be triggered back to V_{sat+} and the threshold moves to V_{th+}. Consequently, the current through R_3 changes sign and the capacitor is charged again. In this way, $v_c(t)$ oscillates between V_{th+} and V_{th-}.

3.3.2 The Period (Frequency) of the Relaxation Oscillator

In what follows, we will calculate the period (frequency) of the relaxation oscillator. We shall ignore the current from and into the negative input pin of Op-Amp; as a consequence, R_3 and C form a first-order circuit.

- Consider the charging process from time 0. Assuming the capacitor is charged but $v_c(t) < V_{th+}$, we have

$$R_3 C v_c'(t) + v_c(t) = v_{out}(t), \tag{11}$$

  where $v_c(0^+) = V_{th-}$ and $v_{out}(t)$ can be treated as a DC Source with $v_{out}(0^+) = V_{sat+}$. Now by (10), if $v_c(t) < V_{th+}$, then

$$v_c(t) = (V_{th-} - V_{sat+})e^{\frac{-t}{R_3 C}} + V_{sat+}. \tag{12}$$

- Consider the discharging process from time 0. Assuming $v_c(t) > V_{th-}$, we have

$$R_3 C v_c'(t) + v_c(t) = v_{out}(t), \tag{13}$$

  where $v_c(0^+) = V_{th+}$ and $V_{out}(0^+) = V_{sat-}$. Hence, if $v_c(t) > V_{th-}$, then

$$v_c(t) = (V_{th+} - V_{sat-})e^{\frac{-t}{R_3 C}} + V_{sat-}. \tag{14}$$

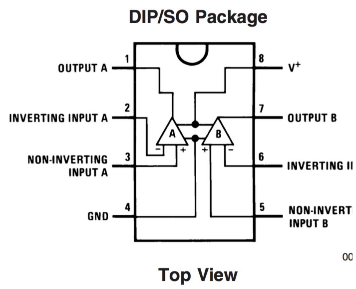

Figure 5: LM358

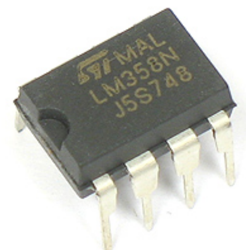

Figure 6: LC358 package

- Denote T_1 (T_2) as the time at which $v_c(t)$ change from V_{th-} (V_{th+}) to V_{th+} (V_{th-}). We have

$$(V_{th-} - V_{sat+})e^{\frac{-T_1}{R_3C}} + V_{sat+} = V_{th+}, \tag{15a}$$

$$(V_{th+} - V_{sat-})e^{\frac{-T_2}{R_3C}} + V_{sat-} = V_{th-}, \tag{15b}$$

which implies

$$T_1 = R_3C \ln \frac{V_{sat+} - V_{th-}}{V_{sat+} - V_{th+}}, \tag{16a}$$

$$T_2 = R_3C \ln \frac{V_{sat-} - V_{th+}}{V_{sat-} - V_{th-}}. \tag{16b}$$

As a result,

$$T = T_1 + T_2 = RC(\ln \frac{V_{sat+} - V_{th-}}{V_{sat+} - V_{th+}} + \ln \frac{V_{sat-} - V_{th+}}{V_{sat-} - V_{th-}}) \tag{17}$$

and $f = \frac{1}{T}$.

4 Experiment

4.1 Preparation:

a. Review of the Schmitt trigger: Assuming that $V_{sat+} = 8.0V$ and $V_{sat-} = -8.0V$, calculate V_{th+} and V_{th-}.

b. Make sure that you are familiar with the connection diagram (the numbering of pins) of LM358.

c. Calculate the period T of the relaxation oscillator in Fig. 4.

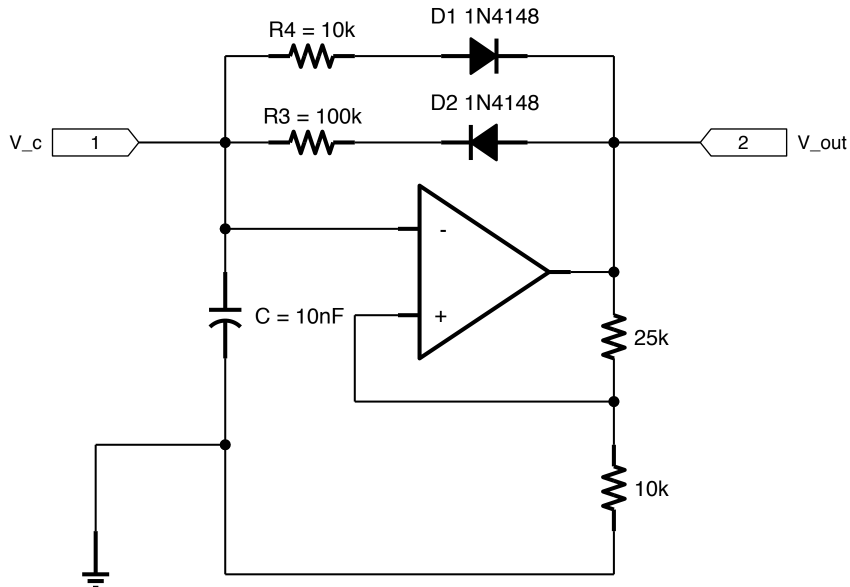

Figure 7: A modified relaxation oscillator

4.2 Experiment:

a. Build the relaxation oscillator according to Fig. 4.

b. Sketch the waveforms of $v_c(t)$ and $v_{out}(t)$ and measure the period (frequency) of $v_{out}(t)$ for $C = 10nF(103)$, $R_3 = 10k\Omega, 100k\Omega$. Repeat the experiment for $C = 100nF(104)$, $R_3 = 10k\Omega, 100k\Omega$.

c. Build the relaxation oscillator according to Fig. 7. Sketch the output waveform and measure the period (frequency). Try to analyze the waveforms of $v_c(t)$ and $v_{out}(t)$ first and find out whether the measurement matches your analysis.

d. *You may play with the circuit by changing R_3 to other values, e.g., $1M\Omega$.

e. *Can you build a circuit by using the other Op-Amp in LM358 to generate a triangular output? Analyze it first and try your best to realize it if you have time.

5 Results and Conclusions

Your report should include the following things:

a. Calculate V_{th+} and V_{th-} by using the measured V_{sat+} and V_{sat-}.

b. Calculate the period (frequency) of the relaxation oscillator in Fig. 4. and compare it with the measurement.

c. Sketch the meansured $v_c(t)$ and $v_{out}(t)$ for different R_3 and C.

d. Assuming the diode 1N4148 is ideal with built-in potential approximately 0.7V, calculate the period of the modified relaxation oscillator in Fig. 7 and also sketch the theoretical $v_c(t)$ and $v_{out}(t)$.