Chapter IV Problems Solutions

1. - Golay's (23,12) codes are three-error-correcting codes. Verify that $n = 23$ and $k = 12$ satisfies the Hamming bound exactly for $t = 3$.
   - Hamming bound: $2^{n-k} \geq \sum_{j=0}^{t} \binom{n}{j}$

$$2^{11} = 2048 \quad \text{and} \quad \sum_{j=0}^{3} \binom{23}{j} = 2048$$

   Therefore, indeed

$$2^{11} \geq \sum_{j=0}^{3} \binom{23}{j}$$

2. • (a) Determine the Hamming bound for a ternary code.

(b) A ternary (11,6) code exists that can correct up to two errors. Verify that this code satisfies the Hamming bound exactly.

• (a) There are $\left(\begin{array}{c} n \\ j \end{array}\right)$ ways to select j positions from n. But for a ternary code, a digit may be mistaken for two other digits. Hence the number of possible errors in j places is

$$\left(\begin{array}{c} n \\ j \end{array}\right)(3-1)^j$$

Thus the Hamming bound is given by

$$3^n \geq 3^k \sum_{j=0}^{t} \left(\begin{array}{c} n \\ j \end{array}\right) 2^j \quad \Longrightarrow \quad 3^{n-k} \geq \sum_{j=0}^{t} \left(\begin{array}{c} n \\ j \end{array}\right) 2^j$$

(b) $3^5 = 243$ and $\sum_{j=0}^{2} \left(\begin{array}{c} n \\ j \end{array}\right) 2^j = 243$

Thus, indeed the Hamming bound found in part (a) is satisfied exactly.

3.

- Confirm the possibility of a (18,7) binary code that can correct up to three errors. Can this code correct up to four errors?
- For (18,7) code to correct up to 3 errors,

$$2^{11} \geq sum_{j=0}^{3} \left(\begin{array}{c} 18 \\ j \end{array} \right)$$

$$\sum_{j=0}^{2} \left(\begin{array}{c} n \\ j \end{array} \right) = 988 \quad \text{and} \quad 2^{11} = 2048$$

Thus the Hamming condition is satisfied and there exists a possibility of 3 error correcting (18,7) code. Since the Hamming distance is over satisfied, this code could correct some 4 error patterns in addition to all patterns with up to 3 errors.

4. • Consider a generator matrix $\mathbf{G}$ for a nonsystematic (6,3) code:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Construct the code for this generator matrix $\mathbf{G}$, and show that d_{min}, the minimum distance between codewords, is 3. Consequently, this code can correct at least one error.

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d			c					
1	1	1	0	1	0	1	1	1
1	1	0	1	0	0	1	1	0
1	0	1	0	1	1	1	0	1
1	0	0	1	0	1	1	0	0
0	1	1	1	1	1	0	1	1
0	1	0	0	0	1	0	1	0
0	0	1	1	1	0	0	0	1
0	0	0	0	0	0	0	0	0

It can be seen that the minimum distance between any two code words is 3. Hence this code can correct at least one error.

5.
- Given a nonsystematic generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

  Construct a (3,1) code. How many errors can this code correct? Find the codeword for data vectors $b = 0$ and $b = 1$.

- $c = d\mathbf{G}$ where d is a single digit (0,1).
  For $d = 0$, $c = 0[1\ 1\ 1] = [0\ 0\ 0]$
  For $d = 1$, $c = 1[1\ 1\ 1] = [1\ 1\ 1]$
  Hence this matrix represents a code that repeats the digit 3 times.
  We have seen earlier that such a code can correct up to 1 error.

6.

- Find a generator matrix **G** for a (15,11) single-error-correcting linear block code. Find the codeword for the data vector 0 1 0 0 1 0 1 0 1 1 1
- H^T is a 15×4 matrix with all distinct rows. One possible H^T is

$$H^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{P} \\ \mathbf{I}_m \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_k & \mathbf{P} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

For

$$d = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$c = d\mathbf{G} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

7. • For a (6,3) systematic linear block code, the three parity-check digits c_4, c_5 and c_6 are

$$c_4 = b_1 + b_2 + b_3$$
$$c_5 = b_1 + b_2$$
$$c_6 = b_2 + b_3$$

(a) Construct the appropriate generator matrix for this code.
(b) Construct the code generated by this matrix.
(c) Determine the error-correcting capabilities of this code.
(d) Prepare a suitable decoding table.
(e) Decode the following received words:101100, 000110, 101010

• (a) It is evident that

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Hence,

$$\mathbf{H}^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $c = d\mathbf{G}$. Hence the code is

d			c					
1	1	1	1	1	1	1	0	0
1	1	0	1	1	0	0	0	1
1	0	1	1	0	1	0	1	1
1	0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1	0
0	1	0	0	1	0	1	1	1
0	0	1	0	0	1	1	0	1
0	0	0	0	0	0	0	0	0

(c) The minimum distance between any two codewords is 3. Hence, this is a single error correcting code. Since there are 6 single errors and 7 syndromes, we can correct all single errors and one double error pattern.

(d) The decoding table is obtained from $S = e\mathbf{H}^T$

S			e					
1	1	0	1	0	0	0	0	0
1	1	1	0	1	0	0	0	0
1	0	1	0	0	1	0	0	0
1	0	0	0	0	0	1	0	0
0	1	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	1
0	1	1	0	0	0	0	1	1

(e) $s = r\mathbf{H}^T$

r						s			e						c						d		
1	0	1	1	0	0	1	1	1	0	1	0	0	0	0	1	1	1	1	0	0	1	1	1
0	0	0	1	1	0	1	1	0	1	0	0	0	0	0	1	0	0	1	1	0	1	0	0
1	0	1	0	1	0	0	0	1	0	0	0	0	0	1	1	0	1	0	1	1	1	0	1

8.

- Construct a single-error-correcting (7,4) linear block code (Hamming Code) and the corresponding decoding table.
- $$\mathbf{G} = \left[\begin{array}{cc} \mathbf{I}_m & \mathbf{P} \end{array}\right] = \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$c = d\mathbf{G}$

d				c						
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	1	1	0
0	0	1	0	0	0	1	0	0	1	1
0	0	1	1	0	0	1	1	1	0	1
0	1	0	0	0	1	0	0	1	1	1
0	1	0	1	0	1	0	1	0	0	1
0	1	1	0	0	1	1	0	1	0	0
0	1	1	1	0	1	1	1	0	1	0
1	0	0	0	1	0	0	0	1	0	1
1	0	0	1	1	0	0	1	0	1	1
1	0	1	0	1	0	1	0	1	1	0
1	0	1	1	1	0	1	1	0	0	0
1	1	0	0	1	1	0	0	0	1	0
1	1	0	1	1	1	0	1	1	0	0
1	1	1	0	1	1	1	0	0	0	1
1	1	1	1	1	1	1	1	1	1	1

$$\mathbf{H}^T = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$s = e\mathbf{H}^T$

e							s		
0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	1	0	0	1	0
0	0	0	0	1	0	0	1	0	0
0	0	0	1	0	0	0	1	1	0
0	0	1	0	0	0	1	0	1	1
0	1	0	0	0	0	0	1	1	1
1	0	0	0	0	0	1	1	0	1

$s = r\mathbf{H}^T$ where $r \equiv$received vector
$c = e \oplus r$ where $c \equiv$corrected code

9. • (a) Given $k = 8$, find the minimum value of n for a code that can correct at least one error.

(b) Choose a generator matrix $\mathbf{G}$ for this code.

(c) How many double errors can this code correct?

(d) Construct a decoding table (syndromes and corresponding correctable error patterns)

• (a) For a single error correcting code, the following condition must be satisfied

$$2^{n-k} \geq n+1 \quad \Longrightarrow \quad 2^{n-8} \geq n+1$$

This condition is satisfied for $n \geq 12$. Choose $n = 12$. This gives a (12,8) code.

(b) $\mathbf{H}^T$ is chosen to have 12 distinct rows of 4 elements with the last 4 rows forming an identity matrix. Hence

$$\mathbf{H}^T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

(c) The number of non-zero syndromes=16-1=15. There are 12 single error patterns. Hence we may be able to correct 3 double-error patterns.

(d)

s				e											
0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0
0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
1	1	1	1	1	0	0	0	0	0	0	1	0	0	0	0
1	1	1	0	0	0	1	0	0	0	0	0	1	0	0	0
1	1	0	1	0	0	0	0	0	0	0	1	0	0	0	1

10.

- (a) Construct a systematic (7,4) cyclic code using the gnerator polynomial $g(x) = x^3 + x + 1$
  (b) What are the error-correcting cpabilities of this code?
  (c) Construct the decoding table
  (d) If the received word is 1101100, determine the transmitted data word.

- (a) $g(x) = x^3 + x + 1$
  For a data sequence, 1111

$$d(x) = x^3 + x^2 + x + 1$$

$$x^3 d(x) = x^6 + x^5 + x^4 + x^3$$

$$\begin{array}{rl} & \underline{x^3 + x^2 + 1} \\ x^3 + x + 1 & | x^6 + x^5 + x^4 + x^3 \\ & \underline{x^6 + \quad + x^4 + x^3} \\ & \quad x^5 \\ & \quad \underline{x^5 + x^3 + x^2} \\ & \qquad x^3 + x^2 \\ & \qquad \underline{x^3 + x + 1} \\ & \qquad \quad x^2 + x + 1 \end{array}$$

$$\Longrightarrow \quad c(x) = (x^3+x^2+1)(x^3+x+1) = x^6+x^5+x^4+x^3+x^2+x+1$$

$$\Longrightarrow \quad \mathbf{c} = 1111111$$

For a data sequence, 1110

$$d(x) = x^3 + x^2 + x$$

$$x^3 d(x) = x^6 + x^5 + x^4$$

$$\begin{array}{rl} & \underline{x^3 + x^2} \\ x^3 + x + 1 & | x^6 + x^5 + x^4 \\ & \underline{x^6 + \quad + x^4 + x^3} \\ & \quad x^5 \quad + x^3 \\ & \quad \underline{x^5 \quad + x^3 + x^2} \\ & \qquad \qquad x^2 \end{array}$$

$$\Longrightarrow \quad c(x) = (x^3 + x^2)(x^3 + x + 1) = x^6 + x^5 + x^4 + x^2$$

$$\Longrightarrow \quad \mathbf{c} = 1110100$$

A similar procedure is used to find the remaining codes (see next table)

d				c						
1	1	1	1	1	1	1	1	1	1	1
1	1	1	0	1	1	1	0	1	0	0
1	1	0	1	1	1	0	1	0	0	1
1	1	0	0	1	1	0	0	0	1	0
1	0	1	1	1	0	1	1	0	0	0
1	0	1	0	1	0	1	0	0	1	1
1	0	0	1	1	0	0	1	1	1	0
1	0	0	0	1	0	0	0	1	0	1
0	1	1	1	0	1	1	1	0	1	0
0	1	1	0	0	1	1	0	0	0	1
0	1	0	1	0	1	0	1	1	0	0
0	1	0	0	0	1	0	0	1	1	1
0	0	1	1	0	0	1	1	1	0	1
0	0	1	0	0	0	1	0	1	1	0
0	0	0	1	0	0	0	1	0	1	1
0	0	0	0	0	0	0	0	0	0	0

(b) From the table it can be seen that the minimum distance between any two codes is 3. Hence, this is a single-error correction code.

(c) There are exactly seven possible nonzero syndromes and thus no double-error patterns can be corrected.
For $\mathbf{e} = 1000000$

$$
\begin{array}{r|l}
 & \overline{x^3 + x + 1} \\
x^3 + x + 1 & x^6 \\
 & \underline{x^6 + x^4 + x^3} \\
 & \quad x^4 + x^3 \\
 & \quad \underline{x^4 \qquad + x^2 + x} \\
 & \qquad x^3 + x^2 + x \\
 & \qquad \underline{x^3 \qquad + x + 1} \\
 & \qquad\quad x^2 \qquad + 1
\end{array}
$$

$$\Longrightarrow \quad s(x) = x^2 + 1 \quad \Longrightarrow \quad \mathbf{s} = 101$$

The remaining syndromes are shown in the following table

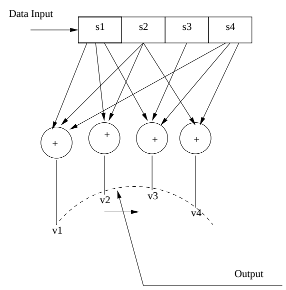
e							s		
1	0	0	0	0	0	0	1	0	1
0	1	0	0	0	0	0	1	1	1
0	0	1	0	0	0	0	1	1	0
0	0	0	1	0	0	0	0	1	1
0	0	0	0	1	0	0	1	0	0
0	0	0	0	0	1	0	0	1	0
0	0	0	0	0	0	1	0	0	1

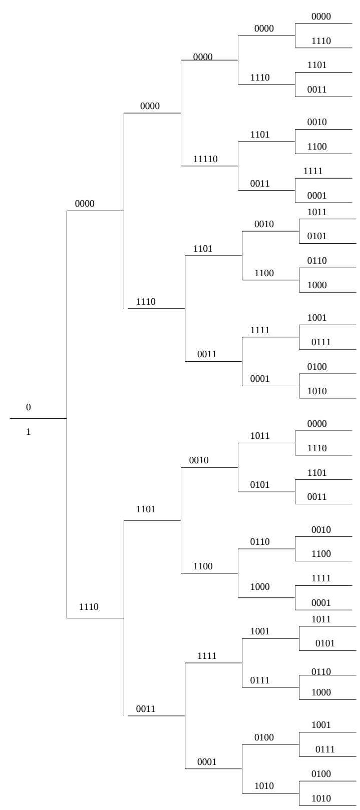
(d) The received data is: 1101100. Thus, $r(x) = x^6 + x^5 + x^3 + x^2$

$$
\begin{array}{r|l}
 & \underline{x^3 + x^2 + x + 1} \\
x^3 + x + 1 & x^6 + x^5 \quad + x^3 + x^2 \\
 & \underline{x^6 \quad + x^4 + x^3} \\
 & \quad x^5 + x^4 \quad + x^2 \\
 & \quad \underline{x^5 \quad + x^3 + x^2} \\
 & \qquad x^4 + x^3 \\
 & \qquad \underline{x^4 \quad + x^2 + x} \\
 & \qquad\quad x^3 + x^2 + x \\
 & \qquad\quad \underline{x^3 \quad x + 1} \\
 & \qquad\qquad x^2 \quad + 1
\end{array}
$$

Therefore, $s(x) = x^2 + 1 \quad \Longrightarrow \quad \mathbf{s} = 101$
From the decoding table, the error corresponding to syndrome $\mathbf{s} = 101$ is $\mathbf{e} = 1000000$. Thus $\mathbf{c} = r \oplus e = 1101100 \oplus 1000000 = 0101100$. Hence, $d = 0101$

11. • Draw the code tree for the convolutional encoder shown in figure and determine the output digit sequence for the data digits 1101011000





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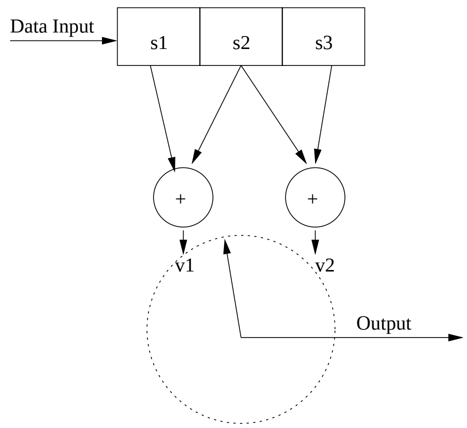
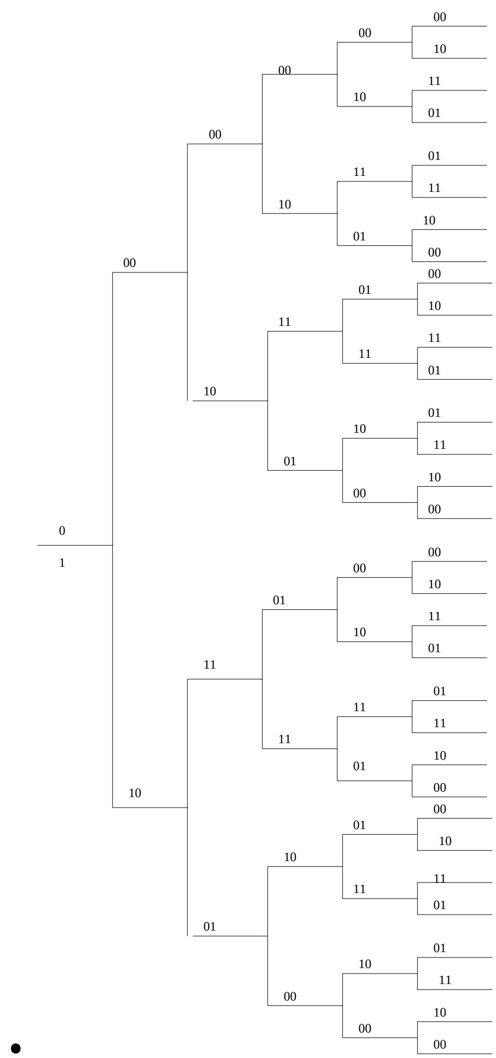
The output digit sequence for 1101011000 is:
1110 0011 1111 0111 0110 1100 1000 1111 1001 1011

12. • Draw the code tree, the trellis diagram. and the state diagram for the convolutional encoder shown in figure.

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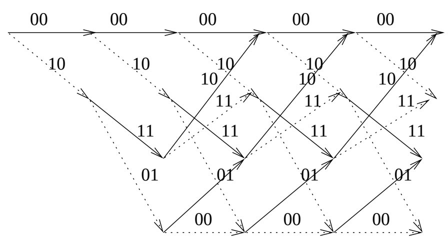

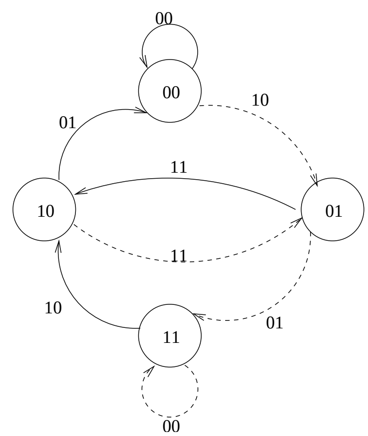
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