Chapter 1: Communications with Digital Signals

Problem 1.1. Determine the range of permissible cutoff frequencies for the ideal lowpass filter used to reconstruct the signal

$$x(t) = 10\cos(600\pi t)\cos^2(1600\pi t)$$

which is sampled at 4000 samples per second. Sketch $X(\omega)$ and $X_s(\omega)$.

Solution:

$$\begin{aligned} x(t) &= 10\cos(600\pi t)\cos^2(1600\pi t) \\ &= 10\cos(600\pi t)[\tfrac{1}{2} + \tfrac{1}{2}\cos(3200\pi t)] \\ &= 5\cos(600\pi t) + 2.5\cos(3800\pi t) + 2.5\cos(2600\pi t) \end{aligned}$$

Sampling frequency = 4000Hz, i.e. spectrum repeats every 4000Hz. The replica of the spectrum are shown dashdotted in Fig. 1.1. The range of the cutoff frequency of the ideal LPF is 1900Hz $< f_c <$ 2100Hz.

$$\text{Note}: \quad \begin{aligned} \cos^2\theta &= \tfrac{1}{2}[1+\cos 2\theta] \\ \cos\theta\cos\phi &= \tfrac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)] \end{aligned}$$

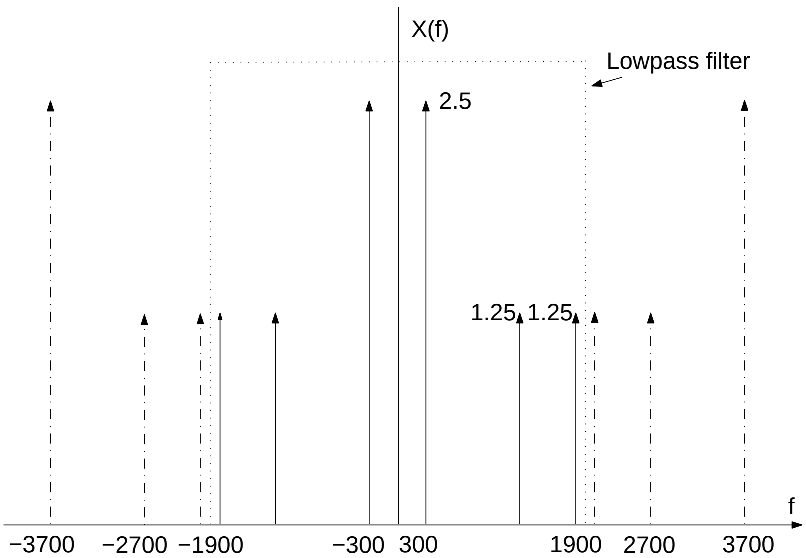


Fig. 1.1 Signal spectrum and its replica

Problem 1.2. Consider a signal $f(t)$ having a probability density function

$$p(f) = \begin{cases} K\mathrm{e}^{-|f|} & -4 < f < 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Find K.

b) Determine the step−size Δ if there are four quantization levels.

c) Calculate the variance of the quantization error when there are four quantization levels. Do not assume that $p(f)$ is constant over each level.

Solution:

a) $$\int_{-\infty}^{\infty} p(f)\mathrm{d}f = \int_{-4}^{4} K\mathrm{e}^{-|f|}\mathrm{d}f = 2K\int_{0}^{4} K\mathrm{e}^{-f}\mathrm{d}f = 2K(1-\mathrm{e}^{-4}) = 1$$

$$\Longrightarrow \quad K = \frac{1}{2(1-\mathrm{e}^{-4})} \simeq 0.51$$

b) The step−size $\Delta = 8/4 = 2$, (quantization levels are $f_1 = -3,\ f_2 = -1,\ f_3 = 1,\ f_4 = 3$).

c) Total variance of the quantization error is equal to the sum of the variance of quantization error at each step of quantization, i.e.

$$\begin{aligned}
\mathrm{E}[e^2] &= \textstyle\sum_{\mathrm{i}=1}^{4} \int_{f_\mathrm{i}-\frac{\Delta}{2}}^{f_\mathrm{i}+\frac{\Delta}{2}} (f-f_\mathrm{i})^2 p(f)\mathrm{d}f \\
&= \textstyle\int_{-4}^{-2} (f-(-3))^2 p(f)\mathrm{d}f + \int_{-2}^{0} (f-(-1))^2 p(f)\mathrm{d}f + \\
&\quad \textstyle + \int_{0}^{2} (f-1)^2 p(f)\mathrm{d}f + \int_{2}^{4} (f-3)^2 p(f)\mathrm{d}f \\
&= \textstyle 2K\int_{0}^{2} (f-1)^2 \mathrm{e}^{-f}\mathrm{d}f + 2K\int_{2}^{4} (f-3)^2 \mathrm{e}^{-f}\mathrm{d}f \\
&= 2K(1-5\mathrm{e}^{-2}) + 2K(\mathrm{e}^{-2}-5\mathrm{e}^{-4}) \\
&\simeq 0.374
\end{aligned}$$

Problem 1.3. A signal $f(t)$ is bandlimited and is ideally sampled at a sampling period T_s such that there is no aliasing error. Each of the ideal samples is then quantized to a step−size Δ. The resulting signal can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t - nT_s) + \sum_{n=-\infty}^{\infty} e_n\delta(t - nT_s)$$

where $f(nT_s)$ are the original unquantized sample values and e_n is the quantization noise associated with the sample at nT_s. Assuming that $\{e_n\}$ is a set of independent random variables, show that the power spectral density of the quantization noise is a constant (white noise) within the frequency range of $-\frac{\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s}$. Also assume that there are many levels of quantization.

Solution:

The noise process is given by

$$e(t) = \sum_{n=-\infty}^{\infty} e_n\delta(t - nT_s)$$

where e_n is a random variable.

We will approach the problem by considering a finite sequence instead of an infinite sequence, i.e. let

$$e_T(t) = \sum_{n=0}^{N-1} e_n\delta(t - nT_s)$$

where N is a finite number. This finite length sequence is shown in Fig. 3.1.

Now imagine a periodic sequence of impulses with period T such that each impulse is of strength e_0. This sequence is shown in Fig 3.2.

The periodic sequence is designated $p_0(t)$ and can be expressed as a Fourier series

$$p_0(t) = e_0(t)\sum_{k=-\infty}^{\infty} \delta(t-kT) = e_0 \sum_{k=-\infty}^{\infty} \alpha_{0k} \exp(jk2\pi t/T) = e_0 \sum_{k=-\infty}^{\infty} \alpha_{0k} \exp(jk\Delta\omega t)$$

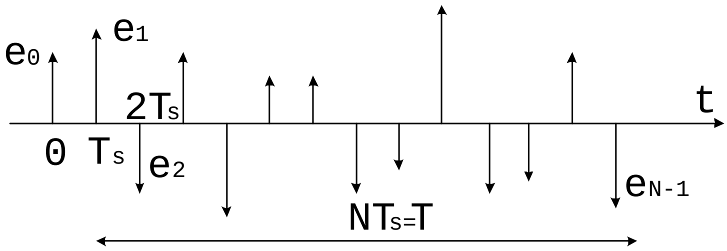


Fig. 3.1 Quantization noise sequence

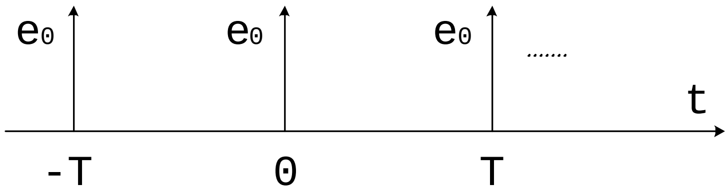


Fig. 3.2 Periodic impulses sequence

where $\Delta\omega = 2\pi/T$. The power of the periodic sequence is given by

$$P_0 = \frac{1}{T}\int_{-T/2}^{T/2} p_0^2(t)\mathrm{d}t = \frac{1}{T}\int_{-T/2}^{T/2} [e_0 \sum_{k=-\infty}^{\infty} \alpha_{0k} \exp(jk\Delta\omega t)]^2 \mathrm{d}t = e_0^2 \sum_{k=-\infty}^{\infty} |\alpha_{0k}|^2$$

But

$$\alpha_{0k} = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) \exp(-jk\Delta\omega t)\mathrm{d}t = \frac{1}{T}, \quad \text{for all} \quad k,$$

Thus P_0 is a staircase function increasing by a step of $\frac{e_0^2}{T^2}$ for every $\Delta\omega$ (Fig. 3.3). The power spectral density of the impulse sequence is given by (see Fig. 3.4)

$$S_{p0}(\omega) = 2\pi\frac{\mathrm{d}P_0}{\mathrm{d}\omega} = 2\pi\frac{e_0^2}{T^2}\sum_{k=-\infty}^{\infty} \delta(\omega - k\Delta\omega)$$

Now consider a similar sequence of impulses each of strength e_1 but occurring at $t = kT + T_s$ (Fig. 3.5).

It can similarly be expressed as a Fourier series such that

$$p_1(t) = e_1 \sum_{k=-\infty}^{\infty} \delta(t - kT - T_s) = e_1 \sum_{k=-\infty}^{\infty} \alpha_{1k} \exp(jk\Delta\omega t - j\Delta\omega T_s)$$

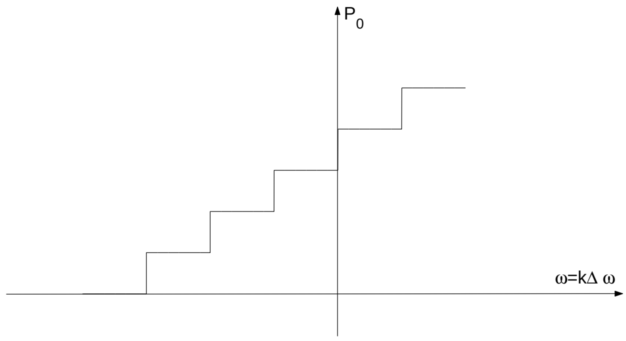


Fig. 3.3 Power of periodic sequence

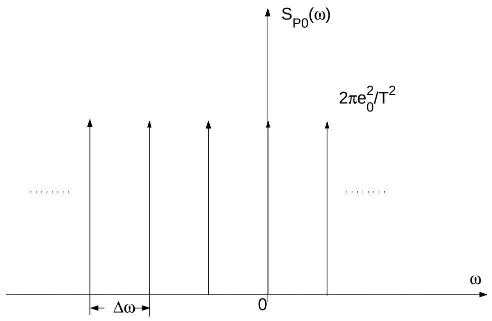


Fig. 3.4 Power spectral density of the impulse sequence

Again $\alpha_{1k} = 1/T$.
Now imagine a periodic sequence $\tilde{e}_T(t)$ of period T, such that the impulses inside each period have strengths $e_0,\ e_1, ..., e_{N-1}$. In other words, imagine the sequence in Fig. 3.1 to repeat itself every $T = NT_s$, then this periodic

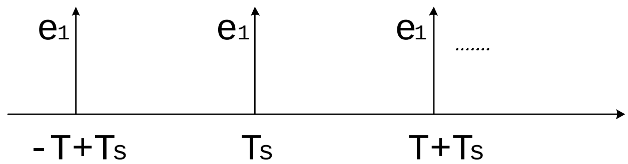


Fig. 3.5 Impulses sequence

sequence can be written as

$$\tilde{e}_T(t) = \sum_{n=0}^{N-1} p_n(t) = \sum_{n=0}^{N-1} e_n \sum_{k=-\infty}^{\infty} \alpha_{nk} \exp(jk\Delta\omega t - j\Delta\omega n T_s)$$
$$= \sum_{n=0}^{N-1} \frac{e_n}{T} \exp(-j\Delta\omega n T_s) \sum_{k=-\infty}^{\infty} \exp(jk\Delta\omega t) \qquad (1)$$

But $\tilde{e}_T(t)$ is a periodic function, therefore it can be expressed as a Fourier series, i.e.

$$\tilde{e}_T(t) = \sum_{k=-\infty}^{\infty} a_k \exp(jk\Delta\omega t) \qquad (2)$$

Comparing (1) and (2), we see that

$$a_k = \sum_{n=0}^{N-1} \frac{e_n}{T} \exp(-j\Delta\omega n T_s) \qquad (3)$$

The power spectral density of $\tilde{e}_T(t)$ is

$$S_{\tilde{e}_T}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} |a_k|^2 \delta(\omega - k\Delta\omega)$$

The power spectral density of $\tilde{e}_T(t)$ associated with the interval is $S_{\tilde{e}_T}(k\Delta\omega)$ such that the power falling within the frequency range $(k\Delta w - \frac{\Delta w}{2})$ to $(k\Delta w + \frac{\Delta\omega}{2})$ is

$$|a_k|^2 = a_k a_k^*$$

But a_k is a random variable since e_n is a random variable. Their relationship is given by Eq. (3). Hence we have to consider the average power falling within the frequency range $(k\Delta\omega - \frac{\Delta\omega}{2})$ to $(k\Delta\omega + \frac{\Delta\omega}{2})$ such that

$$\frac{1}{2\pi} \mathrm{E}\{S_{\tilde{e}_\mathrm{T}}(k\Delta\omega)\Delta\omega\} = \mathrm{E}\{a_k a_k^*\}$$

$$= \mathrm{E}\{\textstyle\sum_{n=0}^{N-1}[\frac{e_n}{T}\exp(-j\Delta\omega n T_s)] \sum_{m=0}^{N-1}[\frac{e_m}{T}\exp(-j\Delta\omega m T_s)]\}$$

But $E[e_n e_m] = 0$ for $m \neq n$ since e_m, e_n are independent and zero mean,

$$
\begin{aligned}
\Longrightarrow \tfrac{1}{2\pi}\mathrm{E}\{S_{\tilde{e}_T}(k\Delta w)\Delta w\} &= \mathrm{E}\{\tfrac{1}{T^2}\textstyle\sum_{n=0}^{N-1} e_n^2 = \tfrac{1}{T^2}\sum_{n=0}^{N-1}\mathrm{E}\{e_n^2\} \\
&= \tfrac{1}{T^2}\textstyle\sum_{n=0}^{N-1}(\tfrac{\Delta^2}{12}) = \tfrac{1}{T^2}N\tfrac{\Delta^2}{12}
\end{aligned}
$$

Now, $T = NT_s$ and $\Delta\omega = 2\pi/T$

$$\Longrightarrow \quad \mathrm{E}\{S_{\tilde{e}_T}(k\Delta\omega)\} = \frac{1}{T_s}\frac{\Delta^2}{12}$$

Finally, we let $T \to \infty$, such that $k\Delta\omega \to \omega$ and we have

$$S_e(\omega) = \lim_{T\to\infty} S_{\tilde{e}_T}(\omega) = \frac{1}{T_s}\frac{\Delta^2}{12}$$

i.e. the noise is white.

Problem 1.4. The signal in Problem 1.3 has probability density uniformly distributed between $\pm V$. It is quantized into M discrete values, i.e.

$$-\left(\frac{M-1}{2}\right)\Delta,\ \ -\left(\frac{M-1}{2}-1\right)\Delta, ..., 0, \Delta, 2\Delta, ..., \left(\frac{M-1}{2}\right)\Delta$$

Find the signal to quantization noise ratio.

Solution:

The ideal sampled signal (without quantization) is given by

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t - nT_s)$$

We are given that $f(nT_s)$ is a random variable evenly distributed between +V and $-$V (Fig. 4.1).

This sampled sequence is very similar to the noise sequence considered in Problem 1.3, except that $f(nT_s)$ has a much larger range than e_n. More precisely, the range for $f(nT_s)$ is from $-\frac{M}{2}\Delta$ to $+\frac{M}{2}\Delta$, while the range for e_n is from $-\frac{\Delta}{2}$ to $+\frac{\Delta}{2}$.

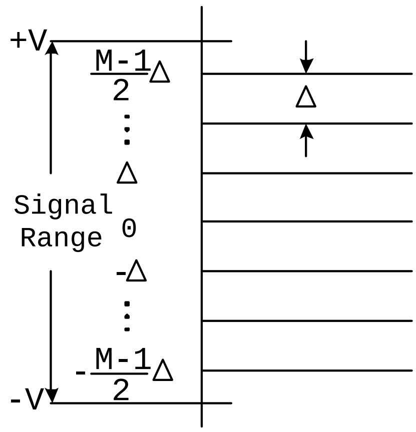


Fig. 4.1 Quantization levels (M levels in total)

Hence we can draw the conclusion that the power spectral density for this sampled signal is given by

$$S_{f_s}(\omega) = \frac{1}{T_s}\frac{(M\Delta)^2}{12} = \frac{1}{T_s}\frac{\mathrm{M}^2\Delta^2}{12}$$

To recover the signal we pass this sequence through a lowpass filter of bandwidth $\omega_c = \omega_s/2 = \pi/T_s$ (Nyquist rate is assumed) so that the signal power at the output of the filter is

$$\frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} S_{f_s}(\omega)\mathrm{d}\omega = \frac{1}{T_s^2}\frac{M^2\Delta^2}{12}$$

The noise power at the output of the filter is

$$\frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} S_e(\omega)\mathrm{d}\omega = \frac{1}{T_s^2}\frac{\Delta^2}{12}$$

$$\Longrightarrow \qquad \left.\frac{S}{N_q}\right|_{\text{output}} = M^2 = 2^{2N}$$

where N is the number of bits used to represent the M levels of quantization.

Problem 1.5. A signal $f(t)$ is not strictly bandlimited. We bandlimit $f(t)$ and then sample it. Due to bandlimiting, distortion occurs even without quantization.

a) Show that the noise power due to the bandlimiting distortion is given by

$$N_D = \frac{1}{\pi}\int_{\omega_m}^{\infty} S_f(\omega)\mathrm{d}\omega$$

where $S_f(\omega)$ is the power spectral density of $f(t)$ and ω_m is the cut$-$off frequency of the bandlimting filter.

b) If $S_f(\omega) = A_0\mathrm{e}^{-|\omega/\omega_0|}$, find N_D.

c) If the bandlimted signal $f(t)$ is sampled at the Nyquist rate and quantized to a step-size Δ, find the total output signal-to-noise power, i.e. find $S_o/(N_D + N_q)$ assuming that the power spectral density of the quantization noise is a constant within $-\pi/\mathrm{T}_s \leq \omega \leq \pi/\mathrm{T}_s$. (See problem 1.3)

Solution:

a) Let the noise power due to the distortion of filtering be N_D, then

$$N_D = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)\mathrm{d}\omega - \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} S_f(\omega)\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\Big[\int_{-\infty}^{-\omega_m} S_f(\omega)\mathrm{d}\omega + \int_{\omega_m}^{\infty} S_f(\omega)\mathrm{d}\omega\Big] = \frac{1}{\pi}\int_{\omega_m}^{\infty} S_f(\omega)\mathrm{d}\omega$$

b) Given that $S_f(\omega) = A_0\mathrm{e}^{-|\omega/\omega_0|}$,

$$\Longrightarrow \quad N_D = \frac{1}{\pi}\int_{\omega_m}^{\infty} A_0\mathrm{e}^{-\omega/\omega_0}\mathrm{d}\omega = \frac{1}{\pi}A_0\omega_0\mathrm{e}^{-\omega_m/\omega_0}$$

c) Let the signal which has been bandlimited to ω_m be $f_B(t)$. This bandlimited signal already consists of a distortion noise $n_D(t)$ the power of which is N_D. Thus we can write

$$f_B(t) = f(t) + n_D(t)$$

The power spectral density of this bandlimited signal is

$$S_{f_B}(\omega) = \begin{cases} A_0 \mathrm{e}^{-|\omega/\omega_0|} & \text{for} \quad -\omega_m \le \omega \le \omega_m \\ 0 & \qquad \text{elsewhere} \end{cases}$$

The bandlimited signal $f_B(t)$ is then sampled at the Nyquist rate, i.e.

$$\omega_s = 2\pi/T_s = 2\omega_m$$

Because of sampling, the power spectral density repeats at every ω_s, and scaled by $\frac{1}{T_s^2}$, i.e.,

$$S_{f_{Bs}}(\omega) = \sum_{n=-\infty}^{\infty} \frac{A_0}{T_s^2} S_{f_B}(\omega - n\omega_s)$$

where $f_{Bs}(t)$ is the bandlimited signal sampled at ω_s. Fig. 5.1 illustrates this fact.

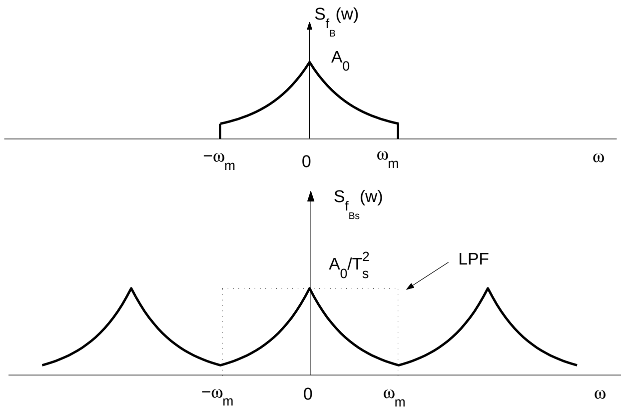


Fig. 5.1 Power spectral density of $f_B(t)$ and $f_{Bs}(t)$

Suppose we do not quantize this sampled bandlimited signal. To recover this signal we use a LPF having cutoff frequency at ω_m. Hence the output signal power is

$$S_o = \frac{1}{2\pi} \int_{-\omega_m}^{\omega_m} \frac{A_0}{T_s^2} \mathrm{e}^{-|\omega/\omega_0|} \mathrm{d}\omega = \frac{1}{\pi} \frac{A_0}{T_s^2} \int_0^{\omega_m} \mathrm{e}^{-\omega/\omega_0} \mathrm{d}\omega$$

$$= \frac{A_0}{\pi T_s^2}\omega_0[1 - \mathrm{e}^{-\omega_m/\omega_0}]$$

Since the signal has been bandlimited before, hence this output signal power is the power of the distorted signal. We can never recover the true signal at the output even if there were no quantization because part of the signal has been filtered off. Hence, we have to regard S_o as the signal power at the output.

¿From Problem 1.3, quantization noise at the output of the filter is

$$N_q = \frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s}\frac{1}{T_s}\frac{\Delta^2}{12}\mathrm{d}\omega = \frac{1}{T_s^2}\frac{\Delta^2}{12}$$

Hence output signal to quantization noise ratio is

$$\frac{S_o}{N_q} = \frac{12A_0}{\pi\Delta^2}w_0[1 - \mathrm{e}^{-w_m/w_0}]$$

Note: N_D has already been taken into account since we chose S_o to be the output signal power.

Problem 1.6. A compressor has the characteristic $f_o = C(f)$ where f is the input signal and f_o is the compressed signal. Thus, is no compression is employed, $f_o = f$.

a) Show that as a result of compression, a uniform step−size of Δ volts in the output f_o results in nonuniform quantization, i.e. varying step−size of the input f. Do this by dividing f_o into 8 equal quantization steps.

b) Show that the variance of the quantization error is now

$$\begin{aligned}\mathrm{E}[\mathrm{e}_c^2] &= \textstyle\int_{f_1-\frac{\Delta_1}{2}}^{f_1+\frac{\Delta_1}{2}}(f-f_1)^2 p(f)\mathrm{d}f + \int_{f_2-\frac{\Delta_2}{2}}^{f_2+\frac{\Delta_2}{2}}(f-f_2)^2 p(f)\mathrm{d}f \\ &\quad + \cdots + \textstyle\int_{f_8-\frac{\Delta_8}{2}}^{f_8+\frac{\Delta_8}{2}}(f-f_8)^2 p(f)\mathrm{d}f\end{aligned}$$

where $f_1 - \frac{\Delta_1}{2} = f_{\min}$ and $f_8 + \frac{\Delta_8}{2} = f_{\max}$, $f_i + \frac{\Delta_i}{2} = f_{i+1} - \frac{\Delta_{i+1}}{2}$.

c) If there are a large number of quantization levels, show that

$$\Delta_i = \frac{\Delta}{C'(f_i)}, \text{ where } C'(f_i) = \left.\frac{\mathrm{d}C(f)}{\mathrm{d}f}\right|_{f=f_i}$$

Hint: Note $\frac{\Delta f_0}{\Delta f_i} \simeq C'(f)$

d) If $p(f)$ is approximately constant throughout each step, show that $\mathrm{E}[\mathrm{e}_c^2]$ becomes

$$\mathrm{E}[\mathrm{e}_c^2] \simeq \frac{1}{12}[\Delta_1^3 p(f_1) + \Delta_2^3 p(f_2) + \cdots + \Delta_8^3 p(f_8)]$$

e) Using the result of c), show that if there are many quantization levels,

$$\mathrm{E}[\mathrm{e}_c^2] = \frac{\Delta^2}{12}\Bigg[\sum_i \frac{\Delta_i p(f_i)}{[C'(f_i)]^2}\Bigg] \simeq \frac{\Delta^2}{12}\int_{\mathrm{f_{imin}}}^{\mathrm{f_{imax}}} \frac{p(f)}{[C'(f)]^2}\mathrm{d}f$$

Solution:

a) See Fig 6.1.

b) From Fig. 6.1,

$$\mathrm{E}[\mathrm{e}_c^2] = \sum_{k=1}^{8}\int_{f_k-\frac{\Delta_k}{2}}^{f_k+\frac{\Delta_k}{2}}(f-f_k)^2 p(f)\mathrm{d}f$$

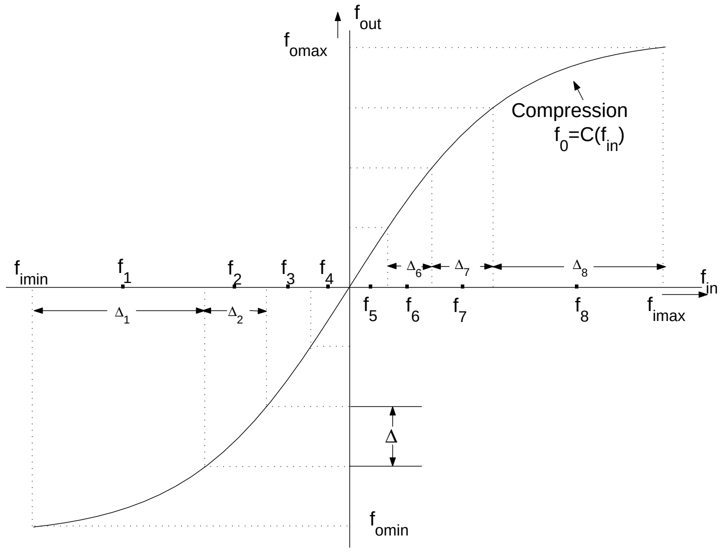


Fig. 6.1 $\mathrm{f_{out}}$ versus $\mathrm{f_{in}}$

c) If there are a large number of quantization levels, then Δ is small,

$$\Longrightarrow \quad \frac{\Delta}{\Delta_k} = \frac{\Delta f_o}{\Delta f_{\text{in}}}\bigg|_{f_{\text{in}}=f_k} \simeq \frac{\mathrm{d}f_0}{\mathrm{d}f_{\text{in}}}\bigg|_{f_{\text{in}}=f_k} = C'(f_{\text{in}})\Big|_{f_{\text{in}}=f_k}$$

$$\Longrightarrow \quad \Delta_k = \frac{\Delta}{C'(f_k)}$$

d)

$$\mathrm{E}[\mathrm{e}_c^2] = \sum_{k=1}^{8} \int_{f_k-\frac{\Delta_k}{2}}^{f_k+\frac{\Delta_k}{2}} (f-f_k)^2 p(f)\mathrm{d}f, \qquad\qquad k = 1, ..., 8$$

But, $p(f) \simeq p(f_k)$ within the interval $f_k - \frac{\Delta_k}{2} \leq f \leq f_k + \frac{\Delta_k}{2}$

$$\Longrightarrow \quad \mathrm{E}[\mathrm{e}_c^2] \simeq \sum_{k=1}^{8} \int_{f_k-\frac{\Delta_k}{2}}^{f_k+\frac{\Delta_k}{2}} (f-f_k)^2 p(f)\mathrm{d}f$$

$$= \sum_{k=1}^{8} p(f_k) \int_{f_k-\frac{\Delta_k}{2}}^{f_k+\frac{\Delta_k}{2}} (f-f_k)^2 \mathrm{d}f = \sum_{k=1}^{8} \frac{\Delta_k^3}{12} p(f_k)$$

e) For small Δ,

$$\mathrm{E}[\mathrm{e}_c^2] \simeq \sum_{k=1}^{8} \frac{\Delta_k^3}{12} p(f_k) \simeq \sum_{k=1}^{8} \frac{\Delta_k}{12} \left[\frac{\Delta}{C'(f)}\right]^2 p(f_k)$$

$$= \frac{\Delta^2}{12} \sum_k \frac{\Delta_k p(f_k)}{[C'(f_k)]^2} \simeq \frac{\Delta^2}{12} \int_{\mathrm{f_{imin}}}^{\mathrm{f_{imax}}} \frac{p(f)}{[C'(f)]^2} \mathrm{d}f$$

Problem 1.7. Find the mean−square quantization error when a signal, $f(t)$, with probability density as shown in Fig. 7.1 is (a) quantized normally, and (b) compressed according to

$$C(f) = \sqrt{|f|}\sin f$$

Assume 4 bit quantization.

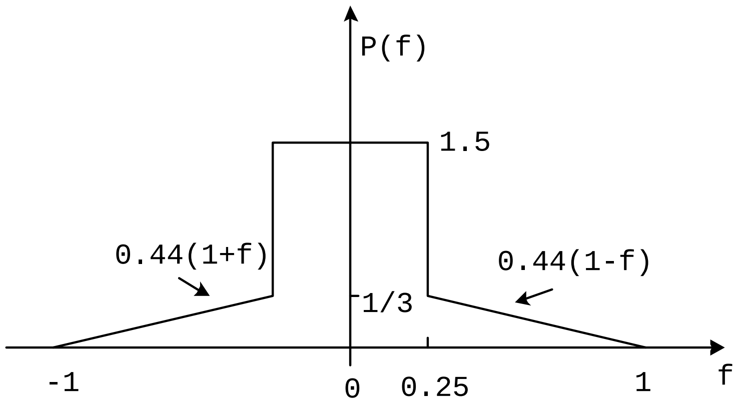


Fig. 7.1 p(f)

Solution:

a) There are 4 bits $\Longrightarrow$ 8 levels of quantization on each side (-1 and 1) of the axis.
Since $f_{\mathrm{imax}} = 1$ and $f_{\mathrm{imin}} = -1$ (from p.d.f. diagram), therefore, each level of quantization is $\frac{1}{8}$ wide, i.e. $f_k - f_{k-1} = \frac{1}{8} = 0.125$, i.e. $\Delta_k = 0.125$, therefore, $f_1 = -0.9375$, $f_2 = -0.8125$, $f_3 = -0.6875$, $\cdots$ $f_{15} = 0.8125$, $f_{16} = 0.9375$.
Similar to part a) of Problem 1.6, we have

$$\mathrm{E}[\mathrm{e}_c^2] = \sum_{k=1}^{16} \int_{f_k - \frac{\Delta_k}{2}}^{f_k + \frac{\Delta_k}{2}} (f - f_k)^2 \mathrm{p}(f)\mathrm{d}f$$

Substituting $\mathrm{p}(f)$ into the different parts of the expression, you should be able to obtain the mean−square error.

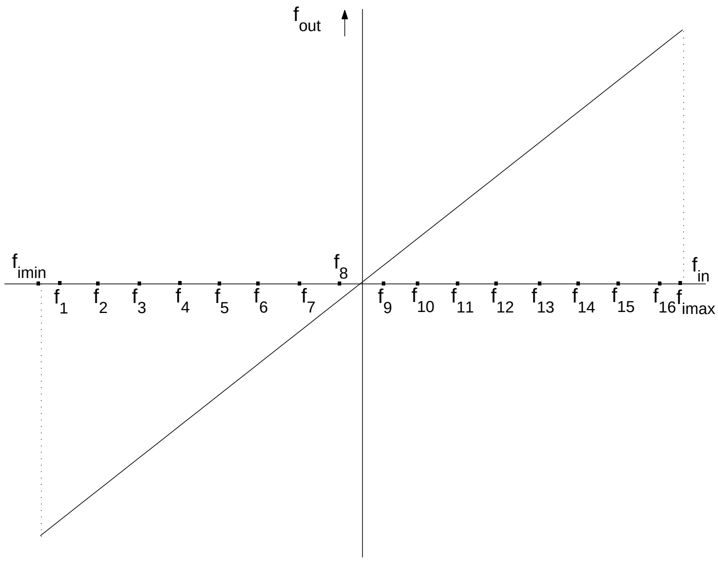


Fig. 7.2 f_{out} versus f_{in}

The detailed solution is very tedious, you can just skip it.

b) Use results in part e) of Problem 1.6 for approximate mean−square error.

Problem 1.8. Assume logarithmic companding with $C(f)$ giving by

$$C(f) = f_{\max}\frac{\log(1+\mu|f|/f_{\max})}{\log(1+\mu)}\mathrm{sgn}(f)$$

where μ is a constant known as the compression parameter. Find the mean square quantization error.
Assume signal to be uniformly probable between $\pm f_{\max}$.

Solution:
Assuming the signal has a pdf uniformly distributed $\pm f_{\max}$, then

$$p(f) = \frac{1}{2f_{\max}}$$

Using part e) of 1.6,

$$\mathrm{E}[\mathrm{e}_c^2] \simeq \frac{\Delta^2}{12}\int_{-\mathrm{f_{max}}}^{\mathrm{f_{max}}} \frac{\frac{1}{2f_{\max}}}{[C'(f)]^2}\mathrm{d}f = \frac{\Delta^2}{12}\int_{0}^{\mathrm{f_{max}}} \frac{\frac{1}{f_{\max}}}{[C'(f)]^2}\mathrm{d}f$$

$$f_{o\max} = f_{\max}\left[\frac{\log(1+\mu)}{\log(1+\mu)}\right] = f_{\max}$$

$$f_{o\min} = -f_{o\max}$$

$$\Delta = (f_{o\max} - f_{o\min})/M = \frac{2f_{\max}}{M}$$

where M is the number of quantization levels.

$$C'(f) = \frac{\mu}{\log(1+\mu)(1+\mu f/f_{\max})}$$

$$\begin{aligned}
\Rightarrow\ \mathrm{E}[\mathrm{e}_c^2] &\simeq \tfrac{\Delta^2}{12}\textstyle\int_0^{\mathrm{f_{max}}} \tfrac{1}{f_{\max}}\tfrac{\log^2(1+\mu)(1+\mu f/f_{\max})^2}{\mu^2}\mathrm{d}f \\
&= \tfrac{f_{\max 2}}{12M^2}\tfrac{1}{\mu^2 f_{\max}}\textstyle\int_0^{\mathrm{f_{max}}} \log^2(1+\mu)(1+\mu f/f_{\max})^2\mathrm{d}f \\
&= \tfrac{f_{\max}^2}{3\mu^2 M^2}[\log(1+\mu)]^2[1+\mu+\tfrac{\mu^2}{3}]
\end{aligned}$$

Problem 1.9. Given an audio waveform

$$f(t) = 3\sin(500t) + 4\sin(1000t) + 4\sin(1500t)$$

find the signal to quantization noise ratio if this is coded using delta modulation.

Solution:

Signal is $f(t) = 3\sin(500t) + 4\sin(1000t) + 4\sin(1500t)$

Therefore signal power is

$$S = \frac{3^2 + 4^2 + 4^2}{2} = 20.5$$

In delta modulation, the sampling period is flexible, the smaller is the sampling period, the smaller is the quantization error. The maximum sampling period is at Nyquist rate, i.e.

$$T_{s\max} = \pi/1500.$$

Now the slope of the signal is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 3\times 500\cos(500t) + 4\times 1000\cos(1000t) + 4\times 1500\cos(1500t)$$

The maximum slope is

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{\max} = 3\times 500 + 4\times 1000 + 4\times 1500 = 11500$$

To ensure no overloading, we have

$$\frac{\Delta}{T_s} \geq \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{\max}, \quad \text{i.e.} \quad \Delta \geq 11500T_s$$

where Δ is the quantization step$-$size. Using the result in Problem 1.4, the quantization noise power is

$$N_q = \frac{\Delta^2\omega_m T_s}{6\pi} = \frac{(11500T_s)^2(1500T_s)}{6\pi}$$

At Nyquist rate,

$$N_q = 11500 \times 11500 \times 1500 \times \frac{\pi^3}{1500^3}/(6\pi) = 96.69$$

Therefore,

$$\frac{S}{N_q} = \frac{20.5}{96.69} = -6.74\text{dB}$$

This value is unacceptable.
Suppose, we increase the sampling rate 32 times, then $T_s = \pi/(32 \times 1500)$, we have

$$N_q = 11500 \times 11500 \times 1500 \times \frac{\pi^3}{1500^3} \times \frac{1}{32^3}/(6\pi) = 2.95 \times 10^{-3}$$

At this sampling rate

$$\frac{S}{N_q} = \frac{20.5}{2.95 \times 10^{-3}} = 38.42\text{dB}$$

which is acceptable.

Problem 1.10. Prove Eq. (1.18) in textbook.

Solution:

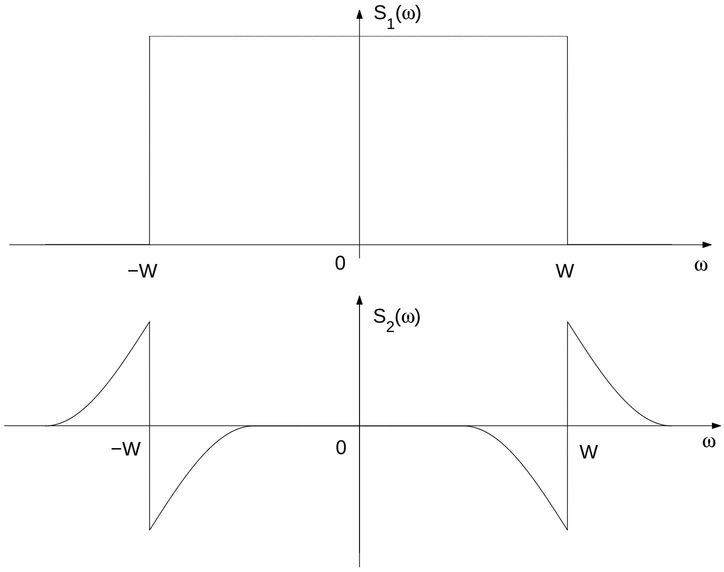


Fig. 10.1 Signal spectrum

$$s_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_1(\omega)\mathrm{e}^{j\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-W}^{W} \mathrm{e}^{j\omega t}\mathrm{d}\omega = \frac{W}{\pi}\left[\frac{\sin(Wt)}{Wt}\right]$$

$$s_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_2(\omega)\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

But $S_2(\omega)$ is an even function. Therefore,

$$\begin{aligned} s_2(t) &= 2[\tfrac{1}{2\pi}\textstyle\int_0^{\infty} S_2(\omega)\cos(\omega t)\mathrm{d}\omega] \\ &= \tfrac{1}{\pi}\textstyle\int_0^{W} S_2(\omega)\cos(\omega t)\mathrm{d}\omega + \tfrac{1}{\pi}\int_W^{2W} S_2(\omega)\cos(\omega t)\mathrm{d}\omega \end{aligned}$$

Let $\omega = W - \lambda$ in the first integral and $\omega = W + \lambda$ in the second integral, then

$$\begin{aligned} s_2(t) &= \tfrac{1}{\pi}\textstyle\int_0^{W} S_2(W-\lambda)\cos(W-\lambda)t\mathrm{d}\lambda + \tfrac{1}{\pi}\int_0^{W} S_2(W+\lambda)\cos(W+\lambda)t\mathrm{d}\lambda \\ &= \tfrac{1}{\pi}\Big[\textstyle\int_0^{W}[S_2(W-\lambda)\cos(W-\lambda)t + S_2(W+\lambda)\cos(W+\lambda)t]\mathrm{d}\lambda\Big] \end{aligned}$$

But $S_2(\omega)$ has odd symmetry about W, i.e. $S_2(W-\lambda) = -S_2(W+\lambda)$, therefore,

$$s_2(t) = \frac{1}{\pi}\Big[\int_0^W S_2(W+\lambda)[\cos(W+\lambda)t - \cos(W-\lambda)t]\mathrm{d}\lambda\Big]$$

Now,

$$\cos(W+\lambda)t - \cos(W-\lambda)t = -2\sin(Wt)\sin(\lambda t)$$

Therefore,

$$s_2(t) = \frac{1}{\pi}\Big[-2\sin Wt\int_0^W S_2(W+\lambda)\sin\lambda t\mathrm{d}\lambda\Big]$$

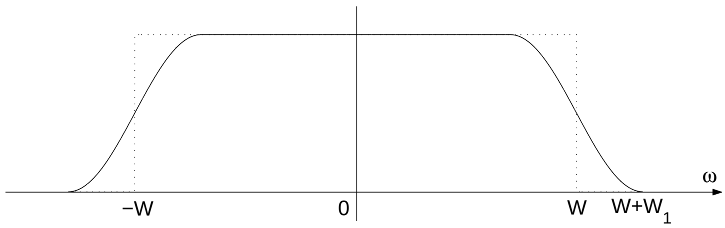
$$= \frac{W}{\pi}\Big[\frac{-\sin Wt}{Wt}2t\int_0^W S_2(W+\omega)\sin\omega t\mathrm{d}\omega\Big]$$

Problem 1.11. A computer output is a train of binary symbols at 56Kbit/sec. Raised−cosine spectral shaping with $W_1/W = 0.3$ is used prior to baseband transmission.

a) Determine the minimum bandwidth required.

b) Repeat if two successive digits are combined into one pulse with four possible amplitudes.

Solution:



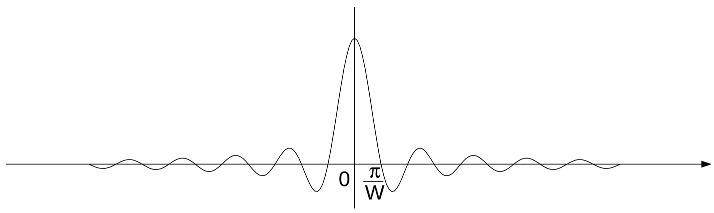


Fig. 11.1

a) Bit period $T_b = \frac{1}{56\times10^3}$, therefore,

$$\frac{\pi}{W} = \frac{1}{56 \times 10^3}$$

Total bandwidth $= W + 0.3W = 1.3W$,

As $W = \pi \times 56 \times 10^3$,

Therefore, total bandwidth$= 1.3\pi \times 56 \times 10^3 = 228.7$Krad/sec.

b) If two pulses are combined into one, the symbol rate is halved. Therefore,

$$\text{symbol period is : } T_s = 2 \times \frac{1}{56 \times 10^3}$$

$$W = \frac{\pi \times 56 \times 10^3}{2}$$

The total bandwidth$= 1.3W = 114.35$Krad/sec.

Problem 1.12. Consider the raised-cosine spectrum of Fig. 1.19(a) in textbook. Assume linear phase shift,

$$\theta(\omega) = -\omega t_0$$

Show that the impulse response signal is given by

$$s(t) = \frac{W}{\pi}\frac{\sin Wt}{Wt}\left[\frac{\cos W_1 t}{1-(2W_1 t/\pi)^2}\right]$$

Solution:

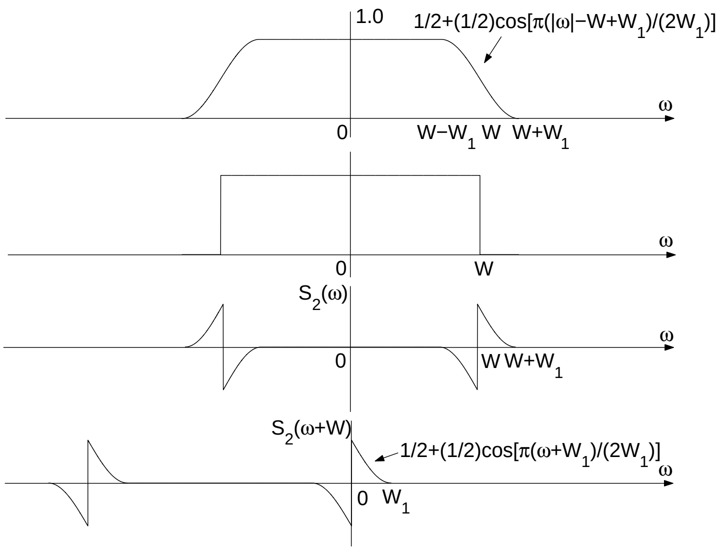


Fig. 12.1

From Eq. (1.19) in textbook,

$$s(t) = \frac{W}{\pi}\frac{\sin Wt}{Wt}\left[1 - 2t\int_0^{W_1} S_2(\omega+W)\sin(\omega t)\mathrm{d}\omega\right]$$

But over the integration interval,

$$S_2(\omega+W) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{\pi}{2W_1}(\omega+W-W+W_1)\right]$$

$$= \frac{1}{2} + \frac{1}{2}\cos\left[\frac{\pi}{2W_1}(\omega + W_1)\right]$$

Therefore,

$$\begin{aligned}
s(t) &= \tfrac{W}{\pi}\tfrac{\sin Wt}{Wt}\Big[1 - 2t\textstyle\int_0^{W_1}\tfrac{1}{2}\sin\omega t + \tfrac{1}{2}\sin\omega t\cos[\tfrac{\pi}{2W_1}(\omega + W_1)]\mathrm{d}\omega\Big] \\
&= \tfrac{W}{\pi}\tfrac{\sin Wt}{Wt}\Big[1 + t\tfrac{\cos\omega t}{t}\Big|_0^{W_1} - \tfrac{t}{2}\textstyle\int_0^{W_1}[\sin(\omega(t + \tfrac{\pi}{2W_1}) + \tfrac{\pi}{2}) + \sin(\omega(t - \tfrac{\pi}{2W_1}) - \tfrac{\pi}{2})]\mathrm{d}\omega\Big] \\
&= \tfrac{W}{\pi}\tfrac{\sin Wt}{Wt}\Big[\cos W_1 t + \tfrac{t}{2}\tfrac{\cos[W_1(t+\frac{\pi}{2W_1})+\frac{\pi}{2}]}{t+\frac{\pi}{2W_1}}\Big|_0^{W_1} + \tfrac{t}{2}\tfrac{\cos[\omega(t-\frac{\pi}{2W_1})-\frac{\pi}{2}]}{t-\frac{\pi}{2W_1}}\Big|_0^{W_1}\Big] \\
&= \tfrac{W}{\pi}\tfrac{\sin Wt}{Wt}\Big[\cos W_1 t - \tfrac{1}{2}\tfrac{\cos W_1 t}{1+\frac{\pi}{2W_1 t}} - \tfrac{1}{2}\tfrac{\cos W_1 t}{1-\frac{\pi}{2W_1 t}}\Big] \\
&= \tfrac{W}{\pi}\tfrac{\sin Wt}{Wt}\cos W_1 t\Big[\tfrac{1-(\frac{\pi}{2W_1 t})^2 - \frac{1}{2}(1-\frac{\pi}{2W_1 t}) - \frac{1}{2}(1+\frac{\pi}{2W_1 t})}{1-(\frac{\pi}{2W_1 t})^2}\Big] \\
&= \tfrac{W}{\pi}\tfrac{\sin Wt}{Wt}\cos W_1 t\Big[\tfrac{(-\frac{\pi}{2W_1 t})^2}{1-(\frac{\pi}{2W_1 t})^2}\Big] \\
&= \tfrac{W}{\pi}\tfrac{\sin Wt}{Wt}\tfrac{\cos W_1 t}{1-(\frac{2W_1 t}{\pi})^2}
\end{aligned}$$

Problem 1.13. Consider a sequence of pulse samples $x(\mathrm{kT}_s)$ which is assumed to have finite energy. The correlation matrix of this sequence is defined as

$$\Phi_{xx} = \mathrm{E}[\mathbf{x}_k \mathbf{x}_k^{\mathrm{T}}]$$

where x_k^{T} is the transpose of x_k, and

$$\mathbf{x}_k(t) = \begin{bmatrix} x(kT_s) \\ x(kT_s - T_s) \\ \vdots \\ x(kT_s - NT_s + T_s) \end{bmatrix}$$

Show that the matrix Φ_{xx} is positive semi$-$definite, i.e. $\mathbf{w}^{\mathrm{T}}\Phi_{xx}\mathbf{w} \geq 0$ for any non$-$zero vector $\mathbf{w}$.

Solution:

$$\Phi_{xx} = \mathrm{E}[\mathbf{x}_k \mathbf{x}_k^{\mathrm{T}}]$$

Therefore,

$$\mathbf{w}^{\mathrm{T}}\Phi_{xx}\mathbf{w} = \mathrm{E}[\mathbf{w}^{\mathrm{T}}\mathbf{x}_k\mathbf{x}_k^{\mathrm{T}}\mathbf{w}] = \mathrm{E}[(\mathbf{w}^{\mathrm{T}}\mathbf{x}_k)(\mathbf{x}_k^{\mathrm{T}}\mathbf{w})]$$

Now,

$$\mathbf{w}^{\mathrm{T}}\mathbf{x}_k = \sum_{n=0}^{N-1} w_n x(kT_s - nT_s) = y(kT_s), (\text{ which is the } k\text{th sample of the output})$$

Also,

$$\mathbf{x}_k^{\mathrm{T}}\mathbf{w} = \sum_{n=0}^{N-1} x(kT_s - nT_s)w_n = y(kT_s)$$

Therefore,

$$\mathbf{w}^{\mathrm{T}}\Phi_{xx}\mathbf{w} = \mathrm{E}[y^2(kT_s)] \geq 0$$

That is Φ_{xx} is positive semi$-$definite.

Problem 1.14. Some radio systems suffer from multipath distortion which is caused by the existence of more than one propagation path between the transmitter and the receiver. Consider a channel the output of which, in response to a signal $s(t)$, is defined by

$$x(t) = K_1 s(t - t_1) + K_2 s(t - t_2)$$

where K_1 and K_2 are constants, and t_1 and t_2 represent transmission delays. It is proposed to use the 3−tap−delay− line filter(Fig. 14.1) to equalize the multipath distortion produced by this channel.

a) Evaluate the transfer function of the channel.

b) Evaluate W_0, W_1 and W_2 in terms of $K_1,\ K_2,\ t_1$and t_2, assuming that $K_2 \ll K_1$ and $t_2 > t_1$.

Solution:

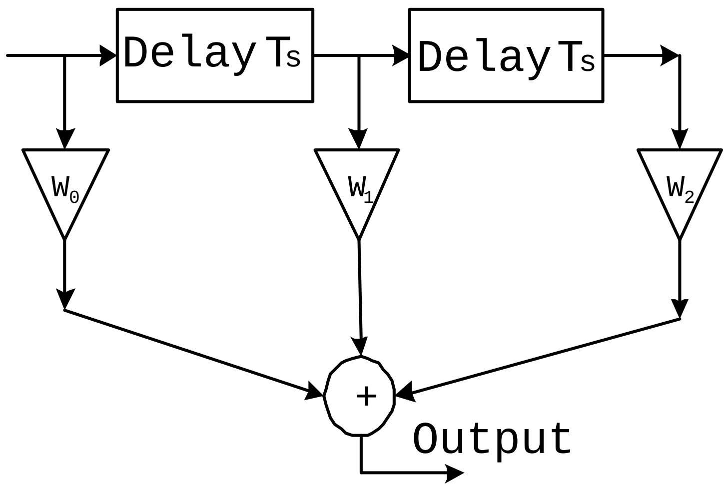


Fig. 14.1

a) The channel output is

$$x(t) = K_1 s(t - t_1) + K_2 s(t - t_2)$$

Taking the Fourier transform, we have

$$X(\omega) = K_1 S(\omega) \mathrm{e}^{-j\omega t_1} + K_2 S(\omega) \mathrm{e}^{-j\omega t_2}$$

Hence the transfer function of the channel is

$$H_c(\omega) = \frac{X(\omega)}{S(\omega)} = K_1 \mathrm{e}^{-j\omega t_1} + K_2(\omega) \mathrm{e}^{-j\omega t_2}$$

b) Ideally, the equalizer should be designed so that

$$H_c(\omega) H_e(\omega) = K_0 \mathrm{e}^{-j\omega t_0}$$

where K_0 and t_0 are constants.
Now for the tap−delay line equalizer, the transfer function is

$$\begin{aligned} H_e(\omega) &= W_0 + W_1 \mathrm{e}^{-j\omega T_s} + W_2 \mathrm{e}^{-j2\omega T_s} \\ &= W_0[1 + \tfrac{W_1}{W_0} \mathrm{e}^{-j\omega T_s} + \tfrac{W_2}{W_0} \mathrm{e}^{-j2\omega T_s}] \end{aligned}$$

For the ideal equalizer,

$$\begin{aligned} H_{e_i}(\omega) &= K_0 \mathrm{e}^{-j\omega t_0} / H_c(\omega) \\ &= K_0 \mathrm{e}^{-j\omega t_0} / [K_1 \mathrm{e}^{-j\omega t_1} + K_2(\omega) \mathrm{e}^{-j\omega t_2}] \\ &= \frac{(K_0/K_1) \mathrm{e}^{-j\omega(t_0 - t_1)}}{1 + (K_2/K_1) \mathrm{e}^{-j\omega(t_2 - t_1)}} \end{aligned} \qquad (4)$$

Using the binomial expansion with $\frac{K_2}{K_1} \ll 1$,

$$H_{e_i}(\omega) \simeq (K_0/K_1) \mathrm{e}^{-j\omega(t_0 - t_1)} [1 - \frac{K_2}{K_1} \mathrm{e}^{-j\omega(t_2 - t_1)} + (\frac{K_2}{K_1})^2 \mathrm{e}^{-j2\omega(t_2 - t_1)} + \ldots] \quad (5)$$

with a 3−tap transversal delay line equalizer, we equate $H_e(\omega)$ to $H_{e_i}(\omega)$ in Eq. (4) and (5). Therefore,

$$\frac{K_0}{K_1} = W_0$$

$$t_0 - t_1 = 0$$

$$-\frac{K_2}{K_1} = \frac{W_1}{W_0}$$

$$\left(\frac{K_2}{K_1}\right)^2 = \frac{W_2}{W_0}$$
$$T_s = t_2 - t_1$$

Choosing $\mathrm{K}_0 = \mathrm{K}_1$, we find that the tap weights are,

$$W_0 = 1, \quad W_1 = -\frac{K_2}{K_1}, \quad W_2 = \left(\frac{K_2}{K_1}\right)^2$$

Problem 1.15. In order to prevent detection error from propagating in the duobinary signalling scheme, we employ the precoding method shown in Fig. 15.1.
We first form the sequence

$$a_k = x_k \oplus a_{k-1}$$

where $\oplus$ represents the modulo-2 sum. Then we obtain the duobinary sequence y_k such that

$$y_k = a_k + a_{k-1} = (x_k \oplus a_{k-1}) + a_{k-1}$$

since $a_k = 0$ or 1, $y_k = 0, 1$ or 2.

a) Find the values of x_k when $y_k = 0$, 1 or 2. Hence obtain a decoding rule at the receiver for $\hat{\mathrm{x}}_k$.

b) If the sequence x_k is (0 0 1 1 0 1 0), find the corresponding sequence a_k, y_k, and $\hat{\mathrm{x}}_k$.

c) Now, assuming an error is made in one of the values of the received sequence y_k, verify that the sequence $\hat{\mathrm{x}}_k$ has only one error in the corresponding position and that the error does not propagate.

Solution:

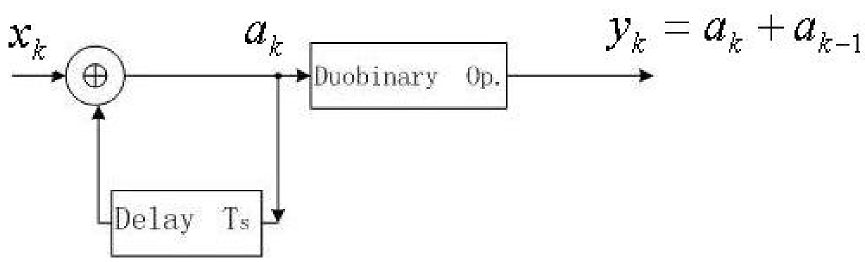


Fig. 15.1 Precoding method

Let x_k be either 0 or 1, then

$$a_k = x_k \oplus a_{k-1}$$

means that $a_k = 0$ or 1, hence

$$y_k = \begin{cases} 0 & \text{if } a_k = 0,\ a_{k-1} = 0 \\ 1 & \text{if } a_k = 1,\ a_{k-1} = 0 \ \text{ or } \ a_k = 0,\ a_{k-1} = 1 \\ 2 & \text{if } a_k = 1,\ a_{k-1} = 1 \end{cases}$$

a)

Case 1: $y_k = 2$

$$\text{Since} \quad y_k = a_k + a_{k-1} = (a_{k-1} \oplus x_k) + a_{k-1}$$

$$y_k = 2 \Rightarrow \begin{cases} a_{k-1} = 1 \\ a_{k-1} \oplus x_k = 1 \Rightarrow x_k = 0 \end{cases}$$

Case 2: $y_k = 0$

$\Rightarrow \quad a_{k-1} = 0$ and $a_k = 0$

$$\Longrightarrow x_k = 0$$

Case 3: $y_k = 1$

$\Rightarrow \quad$ either

$$(1) \quad a_{k-1} = 1 \quad \Rightarrow \quad a_k = 0 \quad \Rightarrow \quad x_k = 1$$

$$(2) \quad a_{k-1} = 0 \quad \Rightarrow \quad a_k = 1 \quad \Rightarrow \quad x_k = 1$$

Hence for

$$y_k = \begin{cases} 0 \\ 2 \end{cases} \quad \Rightarrow \quad x_k = 0$$

and for

$$y_k = 1 \quad \Rightarrow \quad x_k = 1$$

Thus to find $\hat{\mathrm{x}}_k$, we put $\hat{\mathrm{x}}_k = y_k \bmod -2$, i.e., $\hat{\mathrm{x}}_k = y_k$ in binary without carry.

b-c)

$$
\begin{array}{lcccccccc}
x_k & & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
a_k & \overset{\text{assumed}}{1} & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
y_k(\text{no error}) & & 2 & 2 & 1 & 1 & 2 & 1 & 0 \\
\hat{x}_k = y_k \mod -2 & & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
y'_k(\text{with one error}) & & 2 & 1(\text{error}) & 1 & 1 & 2 & 1 & 0 \\
\hat{x}'_k = y'_k \mod -2 & & 0 & \multicolumn{2}{c}{\underbrace{1(\text{error}) \quad 1}_{\text{no propagation}}} & 1 & 0 & 1 & 0
\end{array}
$$

Notice that this precoding scheme resulted in no error propagation. That is error was just limited to one single bit.

Problem 1.16. Show, by using the Shifting Theorem, that Eq. (1.49) in the text book represents the transfer function of the pulse addition network in the duobinary scheme.

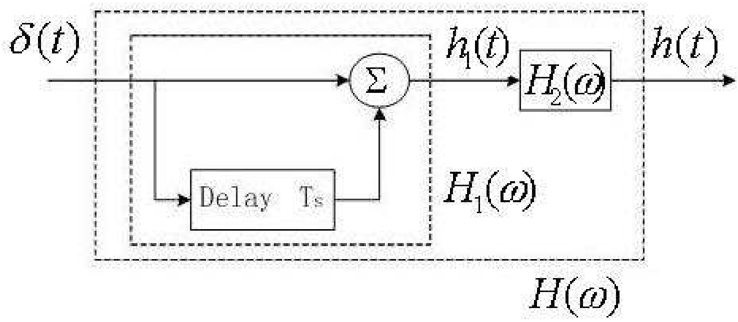


Fig. 16.1

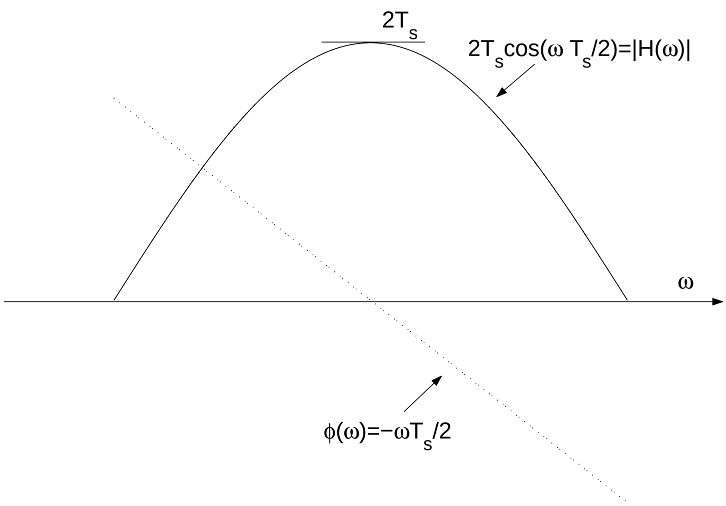


Fig. 16.2

Solution:

$$h_1(t) = \delta(t) + \delta(t - \mathrm{T}_s)$$

Therefore,

$$\begin{aligned} H_1(\omega) &= \int_{-\infty}^{\infty} h_1(t)\mathrm{e}^{-j\omega t}\mathrm{d}t = \int_{-\infty}^{\infty} \delta(t)\mathrm{e}^{-j\omega t}\mathrm{d}t + \int_{-\infty}^{\infty} \delta(t-T_s)\mathrm{e}^{-j\omega t}\mathrm{d}t \\ &= 1 + \mathrm{e}^{-j\omega T_s} \end{aligned}$$

And,

$$H_2(\omega) = \begin{cases} T_s & -\frac{\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$

Therefore,

$$H(\omega) = H_1(\omega)H_2(\omega) = \begin{cases} (1+\mathrm{e}^{-j\omega T_s})T_s = 2T_s\cos(\frac{\omega T_s}{2})\mathrm{e}^{-j\omega T_s/2} & -\frac{\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$

Problem 1.17. Show that in the word synchronization scheme as shown in Fig. 1.26 in the textbook, if each word has n bits excluding the sync bit, and if M of the sync bits are summed together, the probability of having a synchronization error is given by $P_e = 1 - [1 - (1/2^M)]^n$.

Solution:

Let $P_c = \mathrm{P}(\text{correct word sync}) = 1 - P_e$

$P_c = \mathrm{P}$(The 1st bit in each of the M data frames is not equal to 1) $\times$ P(The 2nd bit in each of the M data frames is not equal to 1) $\times \cdots$ P(The nth bit in each of the M data frames is not equal to 1).

Assuming the received bits to be independent and identically distributed, therefore $P_c = \mathrm{P}^n$(The 1st bit in each of the M data frames is not equal to 1), but the probability of occurrence of a specific state in a binary $M-$bit register$= \frac{1}{2^M}$. Thus the probability that this specific state (of all $1'$s) does not occur$= 1 - \frac{1}{2^M}$. Thus,

$$P_c = \left(1 - \frac{1}{2^M}\right)^n$$

Equivalently,

$$P_e = 1 - P_c = 1 - (1 - \frac{1}{2^M})^n$$