

Robust Distributed Compression of Symmetrically Correlated Gaussian Sources

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Abstract—Consider a lossy compression system with ℓ -distributed encoders and a centralized decoder. Each encoder compresses its observed source and forwards the compressed data to the decoder for joint reconstruction of the target signals under the mean-squared-error distortion constraint. It is assumed that the observed sources can be expressed as the sum of the target signals and the corruptive noises, which are generated independently from two symmetric multivariate Gaussian distributions. Depending on the parameters of such distributions, the rate-distortion limit of this system is characterized either completely or at least for sufficiently low distortions. The results are further extended to the robust distributed compression setting, where the outputs of a subset of encoders may also be used to produce a non-trivial reconstruction of the corresponding target signals. In particular, we obtain in the high-resolution regime a precise characterization of the minimum achievable reconstruction distortion based on the outputs of $k + 1$ or more encoders when every k out of all ℓ encoders are operated collectively in the same mode that is greedy in the sense of minimizing the distortion incurred by the reconstruction of the corresponding k target signals with respect to the average rate of these k encoders.

Index Terms—Distributed compression, Gaussian source, Karush-Kuhn-Tucker conditions, mean squared error, rate-distortion.

I. INTRODUCTION

CONSIDER a wireless sensor network where potentially noise-corrupted signals are collected and forwarded to a fusion center for further processing. Due to the communication constraints, it is often necessary to reduce the amount of the transmitted data by local pre-processing at each sensor. Though the multiterminal source coding theory, which aims

to provide a systematic guideline for the implementation of such pre-processing, is far from being complete, significant progress has been made over the past few decades, starting from the seminal work by Slepian and Wolf on the lossless case [1] to the more recent results on the quadratic Gaussian case [2]–[17]. Arguably the greatest insight offered by this theory is that one can capitalize on the statistical dependency among the data at different sites to improve the compression efficiency even when such data need to be compressed in a purely distributed fashion. However, this performance improvement comes at a price: the compressed data from different sites might not be separably decodable, instead they need to be gathered at a central decoder for joint decompression. As a consequence, losing a portion of distributedly compressed data may render the remaining portion completely useless. Indeed, such situations are often encountered in practice. For example, in the aforementioned wireless sensor network, it could happen that the fusion center fails to gather the complete set of compressed data needed for performing joint decompression due to unexpected sensor malfunctions or undesirable channel conditions. A natural question thus arises whether a system can harness the benefits of distributed compression without jeopardizing its functionality in adverse scenarios. Intuitively, there exists a tension between compression efficiency and system robustness. A good distributed compression system should strike a balance between these two factors. The theory intended to characterize the fundamental tradeoff between compression efficiency and system robustness for the centralized setting is known as multiple description coding, which has been extensively studied [18]–[36]. In contrast, its distributed counterpart is far less developed, and the relevant literature is rather scarce [37]–[39].

In the present work we consider a lossy compression system with ℓ distributed encoders and a centralized decoder. Each encoder compresses its observed source and forwards the compressed data to the decoder. Given the data from an arbitrary subset of encoders, the decoder is required to reconstruct the corresponding target signals within a prescribed mean squared error distortion threshold (dependent on the cardinality of that subset). It is assumed that the observed sources can be expressed as the sum of the target signals and the corruptive noises, which are generated independently from two (possibly different) symmetric¹ multivariate Gaussian distributions.

¹This symmetry assumption is not essential for our analysis. It is adopted mainly for the purpose of making the rate-distortion expressions as explicit as possible.

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This setting is similar to that of the robust Gaussian CEO problem studied in [37] and [38]. However, there are two major differences: the robust Gaussian CEO problem imposes the restrictions that 1) the target signal is a scalar process, and 2) the noises across different encoders are independent. Though these restrictions could be justified in certain scenarios, they were introduced largely due to the technical reliance on Oohama's bounding technique for the scalar Gaussian CEO problem [3], [6]. In this paper we shall tackle the more difficult case where the target signals jointly form a vector process by adapting recently developed analytical methods in Gaussian multiterminal source coding theory [10], [13]–[15] to the robust compression setting. Moreover, we show that the theoretical difficulty caused by correlated noises can be circumvented through a fictitious signal-noise decomposition of the observed sources such that the resulting noises are independent across encoders. In fact, it will become clear that this decomposition can be useful even for analyzing those distributed compression systems with independent noises. Our main results are summarized below.

- 1) For the case where the decoder is only required to reconstruct the target signals based on the outputs of all ℓ encoders, the rate-distortion limit is characterized either completely or partially, depending on the parameters of signal and noise distributions,
- 2) For the case where the outputs of a subset of encoders may also be used to produce a non-trivial reconstruction of the corresponding target signals, the minimum achievable reconstruction distortion based on the outputs of $k+1$ or more encoders is characterized either completely or partially, depending on the parameters of signal and noise distributions, when every k out of all ℓ encoders are operated collectively in the same mode that is greedy in the sense of minimizing the distortion incurred by the reconstruction of the corresponding k target signals with respect to the average rate of these k encoders.

The rest of this paper is organized as follows. We state the problem definitions and the main results in Section II. The proofs are presented in Section III and Section IV. We conclude the paper in Section V.

Notation: The expectation operator, the transpose operator, the trace operator, and the determinant operator are denoted by $\mathbb{E}[\cdot]$, $(\cdot)^T$, $\text{tr}(\cdot)$, and $\det(\cdot)$, respectively. A j -dimensional all-one row vector is written as 1_j . We use $\text{diag}^{(j)}(\kappa_1, \dots, \kappa_j)$ to represent a $j \times j$ diagonal matrix with diagonal entries $\kappa_1, \dots, \kappa_j$, and use Y^n as an abbreviation of $(Y(1), \dots, Y(n))$. For a set \mathcal{A} with elements $a_1 < \dots < a_j$, $(\omega_i)_{i \in \mathcal{A}}$ means $(\omega_{a_1}, \dots, \omega_{a_j})$. The cardinality of a set \mathcal{S} is denoted by $|\mathcal{S}|$. Throughout this paper, the base of the logarithm function is e .

II. PROBLEM DEFINITIONS AND MAIN RESULTS

Let the target signals $X \triangleq (X_1, \dots, X_\ell)^T$ and the corruptive noises $Z \triangleq (Z_1, \dots, Z_\ell)^T$ be two mutually independent ℓ -dimensional ($\ell \geq 2$) zero-mean Gaussian random vectors, and the observed sources $S \triangleq (S_1, \dots, S_\ell)^T$ be their sum (i.e., $S = X + Z$). Their respective covariance matrices are

given by

$$\Gamma_X \triangleq \begin{pmatrix} \gamma_X & \rho_X \gamma_X & \cdots & \rho_X \gamma_X \\ \rho_X \gamma_X & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_X \gamma_X \\ \rho_X \gamma_X & \cdots & \rho_X \gamma_X & \gamma_X \end{pmatrix},$$

$$\Gamma_Z \triangleq \begin{pmatrix} \gamma_Z & \rho_Z \gamma_Z & \cdots & \rho_Z \gamma_Z \\ \rho_Z \gamma_Z & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_Z \gamma_Z \\ \rho_Z \gamma_Z & \cdots & \rho_Z \gamma_Z & \gamma_Z \end{pmatrix},$$

$$\Gamma_S \triangleq \begin{pmatrix} \gamma_S & \rho_S \gamma_S & \cdots & \rho_S \gamma_S \\ \rho_S \gamma_S & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_S \gamma_S \\ \rho_S \gamma_S & \cdots & \rho_S \gamma_S & \gamma_S \end{pmatrix},$$

and satisfy $\Gamma_S = \Gamma_X + \Gamma_Z$. Moreover, we construct an i.i.d. process $\{(X(t), Z(t), S(t))\}_{t=1}^\infty$ such that the joint distribution of $X(t) \triangleq (X_1(t), \dots, X_\ell(t))^T$, $Z(t) \triangleq (Z_1(t), \dots, Z_\ell(t))^T$, and $S(t) \triangleq (S_1(t), \dots, S_\ell(t))^T$ is the same as that of X , Z , and S for $t = 1, 2, \dots$.

By the eigenvalue decomposition, every $j \times j$ (real) matrix

$$\Gamma^{(j)} \triangleq \begin{pmatrix} \alpha & \beta & \cdots & \beta \\ \beta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \beta \\ \beta & \cdots & \beta & \alpha \end{pmatrix}$$

can be written as

$$\Gamma^{(j)} = \Theta^{(j)} \Lambda^{(j)} (\Theta^{(j)})^T, \quad (1)$$

where $\Theta^{(j)}$ is an arbitrary (real) unitary matrix with the first column being $\frac{1}{\sqrt{j}} 1_j^T$, and

$$\Lambda^{(j)} \triangleq \text{diag}^{(j)}(\alpha + (j-1)\beta, \alpha - \beta, \dots, \alpha - \beta).$$

For $j \in \{1, \dots, \ell\}$, let $\Gamma_X^{(j)}$, $\Gamma_Z^{(j)}$, and $\Gamma_S^{(j)}$ denote the leading $j \times j$ principal submatrices of Γ_X , Γ_Z , and Γ_S , respectively; in view of (1), we have

$$\Gamma_X^{(j)} = \Theta^{(j)} \Lambda_X^{(j)} (\Theta^{(j)})^T,$$

$$\Gamma_Z^{(j)} = \Theta^{(j)} \Lambda_Z^{(j)} (\Theta^{(j)})^T,$$

$$\Gamma_S^{(j)} = \Theta^{(j)} \Lambda_S^{(j)} (\Theta^{(j)})^T,$$

where

$$\Lambda_X^{(j)} \triangleq \text{diag}^{(j)}(\lambda_{X,1}^{(j)}, \lambda_{X,2}, \dots, \lambda_{X,2}),$$

$$\Lambda_Z^{(j)} \triangleq \text{diag}^{(j)}(\lambda_{Z,1}^{(j)}, \lambda_{Z,2}, \dots, \lambda_{Z,2}),$$

$$\Lambda_S^{(j)} \triangleq \text{diag}^{(j)}(\lambda_{S,1}^{(j)}, \lambda_{S,2}, \dots, \lambda_{S,2})$$

with

$$\begin{aligned}\lambda_{X,1}^{(j)} &\triangleq (1 + (j-1)\rho_X)\gamma_X, \\ \lambda_{X,2} &\triangleq (1 - \rho_X)\gamma_X, \\ \lambda_{Z,1}^{(j)} &\triangleq (1 + (j-1)\rho_Z)\gamma_Z, \\ \lambda_{Z,2} &\triangleq (1 - \rho_Z)\gamma_Z, \\ \lambda_{S,1}^{(j)} &\triangleq (1 + (j-1)\rho_S)\gamma_S, \\ \lambda_{S,2} &\triangleq (1 - \rho_S)\gamma_S.\end{aligned}$$

Note that Γ_X , Γ_Z , and Γ_S are positive semidefinite (and consequently are well-defined covariance matrices) if and only if $\lambda_{X,1}^{(\ell)} \geq 0$, $\lambda_{X,2} \geq 0$, $\lambda_{Z,1}^{(\ell)} \geq 0$, $\lambda_{Z,2} \geq 0$, $\lambda_{S,1}^{(\ell)} \geq 0$, and $\lambda_{S,2} \geq 0$. Furthermore, we assume that $\gamma_X > 0$ since otherwise the target signals are not random. It follows by this assumption that $\gamma_S > 0$, $\lambda_{X,1}^{(\ell)} + \lambda_{X,2} > 0$, and $\lambda_{S,1}^{(\ell)} + \lambda_{S,2} > 0$.

Definition 1: Given $k \in \{1, \dots, \ell\}$, a rate-distortion tuple (r, d_k, \dots, d_ℓ) is said to be achievable if, for any $\epsilon > 0$, there exist encoding functions $\phi_i^{(n)} : \mathbb{R}^n \rightarrow \mathcal{C}_i^{(n)}$, $i = 1, \dots, \ell$, such that

$$\frac{1}{kn} \sum_{i \in \mathcal{A}} \log |\mathcal{C}_i^{(n)}| \leq r + \epsilon, \quad \mathcal{A} \subseteq \{1, \dots, \ell\} \text{ with } |\mathcal{A}| = k, \quad (2)$$

$$\frac{1}{|\mathcal{A}|n} \sum_{i \in \mathcal{A}} \sum_{t=1}^n \mathbb{E}[(X_i(t) - \hat{X}_{i,\mathcal{A}}(t))^2] \leq d_{|\mathcal{A}|} + \epsilon, \quad \mathcal{A} \subseteq \{1, \dots, \ell\} \text{ with } |\mathcal{A}| \geq k, \quad (3)$$

where $\hat{X}_{i,\mathcal{A}}(t) \triangleq \mathbb{E}[X_i(t) | (\phi_{i'}^{(n)}(S_{i'}^n))_{i' \in \mathcal{A}}]$. The set of all such achievable (r, d_k, \dots, d_ℓ) is denoted by \mathcal{RD}_k .

Remark 1: Due to the symmetry of the underlying distributions, it can be shown via a timesharing argument that \mathcal{RD}_k is not affected if we replace (2) with either of the following constraints

$$\begin{aligned}\frac{1}{n} \log |\mathcal{C}_i^{(n)}| &\leq r + \epsilon, \quad i = 1, \dots, \ell, \\ \frac{1}{\ell n} \sum_{i=1}^{\ell} \log |\mathcal{C}_i^{(n)}| &\leq r + \epsilon,\end{aligned}$$

and/or replace (3) with either of the following constraints

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n \mathbb{E}[(X_i(t) - \hat{X}_{i,\mathcal{A}}(t))^2] &\leq d_{|\mathcal{A}|} + \epsilon, \\ \mathcal{A} &\subseteq \{1, \dots, \ell\} \text{ with } |\mathcal{A}| \geq k, \\ \frac{1}{\binom{n}{j} j n} \sum_{\mathcal{A} \subseteq \{1, \dots, \ell\}; |\mathcal{A}|=j} \sum_{i \in \mathcal{A}} \sum_{t=1}^n \mathbb{E}[(X_i(t) - \hat{X}_{i,\mathcal{A}}(t))^2] &\leq d_j + \epsilon, \\ j &= k, \dots, \ell.\end{aligned}$$

Remark 2: We show in Appendix A that, for $j = k, \dots, \ell$,

$$\begin{aligned}d_{\min}^{(j)} &\triangleq \frac{1}{j} \sum_{i=1}^j \mathbb{E}[(X_i - \mathbb{E}[X_i | S_1, \dots, S_j])^2] \\ &= \frac{1}{j} d_{\min,1}^{(j)} + \frac{j-1}{j} d_{\min,2},\end{aligned}$$

where

$$d_{\min,1}^{(j)} \triangleq \begin{cases} 0, & \lambda_{S,1}^{(j)} = 0, \\ \frac{\lambda_{X,1}^{(j)} \lambda_{Z,1}^{(j)}}{\lambda_{S,1}^{(j)}}, & \text{otherwise,} \end{cases}$$

$$d_{\min,2} \triangleq \begin{cases} 0, & \lambda_{S,2} = 0, \\ \frac{\lambda_{X,2} \lambda_{Z,2}}{\lambda_{S,2}}, & \text{otherwise.} \end{cases}$$

It is clear that $d_j > d_{\min}^{(j)}$, $j = k, \dots, \ell$, for any $(r, d_k, \dots, d_\ell) \in \mathcal{RD}_k$. Moreover, if $d_j \geq \gamma_X$ for some $j \in \{k, \dots, \ell\}$, then the corresponding distortion constraint is redundant. Henceforth we shall focus on the case $d_j \in (d_{\min}^{(j)}, \gamma_X)$, $j = k, \dots, \ell$.

Definition 2: For $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$, let

$$r^{(\ell)}(d_\ell) \triangleq \min\{r : (r, d_\ell) \in \mathcal{RD}_\ell\}.$$

In order to state our main results, we introduce the following quantities. For any $k \in \{1, \dots, \ell\}$ and $d_k \in (d_{\min}^{(k)}, \gamma_X)$, let

$$\begin{aligned}\bar{r}^{(k)}(d_k) &\triangleq \frac{1}{2k} \log \frac{(\lambda_{S,1}^{(k)} + \lambda_Q^{(k)})(\lambda_{S,2} + \lambda_Q^{(k)})^{k-1}}{(\lambda_Q^{(k)})^k}, \\ d_j^{(k)}(d_k) &\triangleq \frac{\lambda_{X,1}^{(j)}(\lambda_{Z,1}^{(j)} + \lambda_Q^{(k)})}{j(\lambda_{S,1}^{(j)} + \lambda_Q^{(k)})} \\ &\quad + \frac{(j-1)\lambda_{X,2}(\lambda_{Z,2} + \lambda_Q^{(k)})}{j(\lambda_{S,2} + \lambda_Q^{(k)})}, \quad j = k, \dots, \ell,\end{aligned}$$

where $\lambda_Q^{(k)}$ is the unique positive number satisfying

$$\frac{\lambda_{X,1}^{(k)}(\lambda_{Z,1}^{(k)} + \lambda_Q^{(k)})}{k(\lambda_{S,1}^{(k)} + \lambda_Q^{(k)})} + \frac{(k-1)\lambda_{X,2}(\lambda_{Z,2} + \lambda_Q^{(k)})}{k(\lambda_{S,2} + \lambda_Q^{(k)})} = d_k. \quad (4)$$

Our first result is a partial characterization of $r^{(\ell)}(d_\ell)$.

Theorem 1: For $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$,

$$r^{(\ell)}(d_\ell) = \bar{r}^{(\ell)}(d_\ell)$$

if either of the following conditions is satisfied:

1) $\rho_S \geq 0$ and

$$(\ell-1)\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2\mu^{(\ell)}(\mu^{(\ell)} - 1) + \ell(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2 \geq 0, \quad (5)$$

where

$$\mu^{(\ell)} \triangleq \frac{\lambda_{S,2} - \lambda_{S,2}^2(\lambda_{S,2} + \lambda_Q^{(\ell)})^{-1}}{\lambda_{S,1}^{(\ell)} - (\lambda_{S,1}^{(\ell)})^2(\lambda_{S,1}^{(\ell)} + \lambda_Q^{(\ell)})^{-1}}. \quad (6)$$

2) $\rho_S \leq 0$ and

$$(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2\nu^{(\ell)}(\nu^{(\ell)} - 1) + \ell\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2 \geq 0, \quad (7)$$

where

$$\nu^{(\ell)} \triangleq \frac{\lambda_{S,1}^{(\ell)} - (\lambda_{S,1}^{(\ell)})^2(\lambda_{S,1}^{(\ell)} + \lambda_Q^{(\ell)})^{-1}}{\lambda_{S,2} - \lambda_{S,2}^2(\lambda_{S,2} + \lambda_Q^{(\ell)})^{-1}}. \quad (8)$$

Remark 3: 1) Consider the case $\rho_S \geq 0$. When $(\ell-1)\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2 \leq 4\ell(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2$, the inequality (5) always holds, and $r^{(\ell)}(d_\ell)$ is characterized for all $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$. When $(\ell-1)\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2 > 4\ell(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2$,

the equation $(\ell - 1)\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2\mu^{(\ell)}(\mu^{(\ell)} - 1) + \ell(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2 = 0$ has two real roots in the interval $[0, 1]$:

$$\mu_1^{(\ell)} \triangleq \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4\ell(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2}{(\ell - 1)\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2}},$$

$$\mu_2^{(\ell)} \triangleq \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\ell(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2}{(\ell - 1)\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2}}.$$

Therefore, the inequality (5) holds if

$$\mu^{(\ell)} \leq \mu_1^{(\ell)} \quad \text{or} \quad \mu^{(\ell)} \geq \mu_2^{(\ell)}. \quad (9)$$

It is easy to verify that (9) is satisfied when $\lambda_{S,1}^{(\ell)} > \lambda_{S,2} = 0$ (which implies $\mu^{(\ell)} = 0$) or $\lambda_{S,1}^{(\ell)} = \lambda_{S,2} > 0$ (which implies $\mu^{(\ell)} = 1$). When $\lambda_{S,1}^{(\ell)} > \lambda_{S,2} > 0$, $\mu^{(\ell)}$ is a strictly decreasing function of d_ℓ , converging to 1 as $d_\ell \rightarrow d_{\min}^{(\ell)}$ and to $\frac{\lambda_{S,2}}{\lambda_{S,1}^{(\ell)}}$ as $d_\ell \rightarrow \gamma_X$; hence, it suffices to analyze the following four scenarios.

- $\mu_2^{(\ell)} \leq \frac{\lambda_{S,2}}{\lambda_{S,1}^{(\ell)}}$: $\mu^{(\ell)} \geq \mu_2^{(\ell)}$ is satisfied for all $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$.
- $\mu_1^{(\ell)} \leq \frac{\lambda_{S,2}}{\lambda_{S,1}^{(\ell)}}$ and $\frac{\lambda_{S,2}}{\lambda_{S,1}^{(\ell)}} < \mu_2^{(\ell)} < 1$: $\mu^{(\ell)} \geq \mu_2^{(\ell)}$ is satisfied for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$.
- $\mu_1^{(\ell)} > \frac{\lambda_{S,2}}{\lambda_{S,1}^{(\ell)}}$ and $\mu_2^{(\ell)} < 1$: $\mu^{(\ell)} \leq \mu_1^{(\ell)}$ is satisfied for all d_ℓ sufficiently close to γ_X while $\mu^{(\ell)} \geq \mu_2^{(\ell)}$ is satisfied for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$.
- $\mu_1^{(\ell)} = 0$ and $\mu_2^{(\ell)} = 1$: This can happen only when $\lambda_{X,1}^{(\ell)} = 0$.

In view of the above discussion, under the condition $\rho_S \geq 0$, $r^{(\ell)}(d_\ell)$ is characterized at least for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$ unless $\lambda_{X,1}^{(\ell)} = 0$ and $\lambda_{S,1}^{(\ell)} > \lambda_{S,2}$ (note that $\lambda_{X,1}^{(\ell)} = 0$ implies $\lambda_{S,2} > 0$).

- Consider the case $\rho_S \leq 0$. When $(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2 \leq 4\ell\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2$, the inequality (7) always holds, and $r^{(\ell)}(d_\ell)$ is characterized for all $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$. When $(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2 > 4\ell\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2$, the equation $(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2\nu^{(\ell)}(\nu^{(\ell)} - 1) + \ell\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2 = 0$ has two real roots in the interval $[0, 1]$:

$$\nu_1^{(\ell)} \triangleq \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4\ell\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2}{(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2}},$$

$$\nu_2^{(\ell)} \triangleq \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\ell\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2}{(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2}}.$$

Therefore, the inequality (7) holds if

$$\nu^{(\ell)} \leq \nu_1^{(\ell)} \quad \text{or} \quad \nu^{(\ell)} \geq \nu_2^{(\ell)}. \quad (10)$$

It is easy to verify that (10) is satisfied when $\lambda_{S,2} > \lambda_{S,1}^{(\ell)} = 0$ (which implies $\nu^{(\ell)} = 0$) or $\lambda_{S,1}^{(\ell)} = \lambda_{S,2} > 0$ (which implies $\nu^{(\ell)} = 1$). When $\lambda_{S,2} > \lambda_{S,1}^{(\ell)} > 0$, $\nu^{(\ell)}$ is a strictly decreasing function of d_ℓ , converging to 1

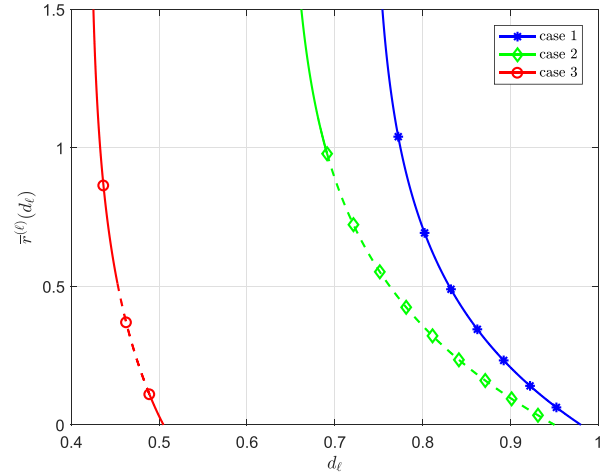


Fig. 1. Plots of $\bar{r}^{(\ell)}(d_\ell)$ with $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$ for the three cases in Example 1.

as $d_\ell \rightarrow d_{\min}^{(\ell)}$ and to $\frac{\lambda_{S,1}^{(\ell)}}{\lambda_{S,2}}$ as $d_\ell \rightarrow \gamma_X$; hence, it suffices to analyze the following four scenarios.

- $\nu_2^{(\ell)} \leq \frac{\lambda_{S,1}^{(\ell)}}{\lambda_{S,2}}$: $\nu^{(\ell)} \geq \nu_2^{(\ell)}$ is satisfied for all $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$.
- $\nu_1^{(\ell)} \leq \frac{\lambda_{S,1}^{(\ell)}}{\lambda_{S,2}}$ and $\frac{\lambda_{S,1}^{(\ell)}}{\lambda_{S,2}} < \nu_2^{(\ell)} < 1$: $\nu^{(\ell)} \geq \nu_2^{(\ell)}$ is satisfied for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$.
- $\nu_1^{(\ell)} > \frac{\lambda_{S,1}^{(\ell)}}{\lambda_{S,2}}$ and $\nu_2^{(\ell)} < 1$: $\nu^{(\ell)} \leq \nu_1^{(\ell)}$ is satisfied for all d_ℓ sufficiently close to γ_X while $\nu^{(\ell)} \geq \nu_2^{(\ell)}$ is satisfied for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$.
- $\nu_1^{(\ell)} = 0$ and $\nu_2^{(\ell)} = 1$: This can happen only when $\lambda_{X,2} = 0$.

In view of the above discussion, under the condition $\rho_S \leq 0$, $r^{(\ell)}(d_\ell)$ is characterized at least for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$ unless $\lambda_{X,2} = 0$ and $\lambda_{S,2} > \lambda_{S,1}^{(\ell)}$ (note that $\lambda_{X,2} = 0$ implies $\lambda_{S,1}^{(\ell)} > 0$).

Example 1: Let $\ell = 10$. In Fig. 1 we plot $\bar{r}^{(\ell)}(d_\ell)$ with $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$ for the following three cases, all with $\rho_S > 0$.

- Case 1:** $\lambda_{X,1}^{(\ell)} = 0.8$, $\lambda_{X,2} = 1$, $\lambda_{S,1}^{(\ell)} = 5$, and $\lambda_{S,2} = 4$. In this case $d_{\min}^{(\ell)} = 0.7422$ and $\gamma_X = 0.98$. It can be verified that $(\ell - 1)\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2 = 225 < 4\ell(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2 = 409.6$. Therefore, $r^{(\ell)}(d_\ell)$ coincides with $\bar{r}^{(\ell)}(d_\ell)$ for all $d_\ell \in (d_{\min}^{(\ell)}, \gamma_X)$.
- Case 2:** $\lambda_{X,1}^{(\ell)} = 0.5$, $\lambda_{X,2} = 1$, $\lambda_{S,1}^{(\ell)} = 6$, and $\lambda_{S,2} = 3$. In this case $d_{\min}^{(\ell)} \approx 0.6458$ and $\gamma_X = 0.95$. It can be verified that $\mu_1^{(\ell)} \approx 0.0751 < \frac{\lambda_{S,2}}{\lambda_{S,1}^{(\ell)}} = 0.5 < \mu_2^{(\ell)} \approx 0.9249 < 1$. Therefore, $r^{(\ell)}(d_\ell)$ coincides with $\bar{r}^{(\ell)}(d_\ell)$ for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$ (see the solid-line portion).
- Case 3:** $\lambda_{X,1}^{(\ell)} = 1$, $\lambda_{X,2} = 0.45$, $\lambda_{S,1}^{(\ell)} = 12$, and $\lambda_{S,2} = 2.4$. In this case $d_{\min}^{(\ell)} = 0.4207$ and $\gamma_X = 0.505$. It can be verified that $\mu_1^{(\ell)} \approx 0.3253 > \frac{\lambda_{S,2}}{\lambda_{S,1}^{(\ell)}} = 0.2$ and $\mu_2^{(\ell)} \approx 0.6747 < 1$. Therefore, $r^{(\ell)}(d_\ell)$ coincides with $\bar{r}^{(\ell)}(d_\ell)$ for all d_ℓ sufficiently close to $d_{\min}^{(\ell)}$ and all d_ℓ sufficiently close to γ_X (see the solid-line portion).

Theorem 1 is a special case of the following more general result.

Theorem 2: 1) For $d_k \in (d_{\min}^{(k)}, \gamma_X)$,

$$(\bar{r}^{(k)}(d_k), d_k^{(k)}(d_k), \dots, d_\ell^{(k)}(d_k)) \in \mathcal{RD}_k.$$

2) For $(r, d_k, \dots, d_\ell) \in \mathcal{RD}_k$ with $d_k \in (d_{\min}^{(k)}, \gamma_X)$,

$$r \geq \bar{r}^{(k)}(d_k)$$

if either of the following conditions is satisfied:

i) $\rho_S \geq 0$ and

$$(k-1)\lambda_{X,2}^2(\lambda_{S,1}^{(k)})^2\mu^{(k)}(\mu^{(k)}-1) + k(\lambda_{X,1}^{(k)})^2\lambda_{S,2}^2 \geq 0, \quad (11)$$

where $\mu^{(k)}$ is defined in (6) with ℓ replaced by k .

ii) $\rho_S \leq 0$ and

$$(\lambda_{X,1}^{(k)})^2\lambda_{S,2}^2\nu^{(k)}(\nu^{(k)}-1) + k\lambda_{X,2}^2(\lambda_{S,1}^{(k)})^2 \geq 0, \quad (12)$$

where $\nu^{(k)}$ is defined in (8) with ℓ replaced by k .

3) For $j \in \{k, \dots, \ell\}$ and $(r, d_k, \dots, d_\ell) \in \mathcal{RD}_k$ with $d_k \in (d_{\min}^{(k)}, \gamma_X)$ and $r = \bar{r}^{(k)}(d_k)$, we have

$$d_j \geq d_j^{(k)}(d_k)$$

if either of the following conditions is satisfied:

a) Condition i).

b) $\rho_S \leq 0$, $\lambda_{S,1}^{(j)} > 0$, and

$$(\nu^{(k,j)} + (k-1))(\lambda_{X,1}^{(k)})^2\lambda_{S,2}^2(\nu^{(k)})^2 + (k-1)(\nu^{(k,j)} - \nu^{(k)})\lambda_{X,2}^2(\lambda_{S,1}^{(k)})^2 \geq 0, \quad (13)$$

$$(\nu^{(k,j)} - 1)(\lambda_{X,1}^{(k)})^2\lambda_{S,2}^2(\nu^{(k)})^2 + ((k-1)\nu^{(k,j)} + \nu^{(k)})\lambda_{X,2}^2(\lambda_{S,1}^{(k)})^2 \geq 0, \quad (14)$$

where

$$\nu^{(k,j)} = \frac{\lambda_{S,1}^{(j)} - (\lambda_{S,1}^{(j)})^2(\lambda_{S,1}^{(j)} + \lambda_Q^{(k)})^{-1}}{\lambda_{S,2} - \lambda_{S,2}^2(\lambda_{S,2} + \lambda_Q^{(k)})^{-1}}.$$

Proof: See Section III for the proof of the achievability part (i.e., Part 1) of Theorem 2) and Section IV for the proof of the converse part (i.e., Part 2) and Part 3) of Theorem 2). ■

Remark 4: 1) The argument in Remark 3 can be leveraged to prove that, for the case $\rho_S \geq 0$, the inequality (11) holds at least for all d_k sufficiently close to $d_{\min}^{(k)}$ unless $\lambda_{X,1}^{(k)} = 0$ (which can happen only when $k = \ell$) and $\lambda_{S,1}^{(k)} > \lambda_{S,2}$ (note that $\lambda_{X,1}^{(k)} = 0$ implies $\lambda_{S,2} > 0$); similarly, for the case $\rho_S \leq 0$, the inequality (12) holds at least for all d_k sufficiently close to $d_{\min}^{(k)}$ unless $\lambda_{X,2} = 0$ and $\lambda_{S,2} > \lambda_{S,1}^{(k)}$ (note that $\lambda_{X,2} = 0$ implies $\lambda_{S,1}^{(k)} > 0$).

2) For the case $\rho_S \leq 0$, the condition $\lambda_{S,1}^{(j)} > 0$ can be potentially violated (i.e., $\lambda_{S,1}^{(j)} = 0$) only when $j = \ell$.

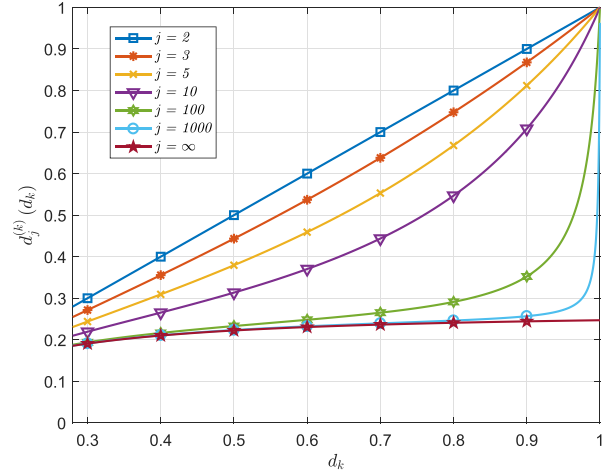


Fig. 2. Plots of $d_j^{(k)}(d_k)$ with $d_k \in (d_{\min}^{(k)}, \gamma_X)$ for $j = 2, 3, 5, 10, 100, 1000, \infty$, where the relevant parameters are specified in Example 2.

3) Consider the case $\rho_S \leq 0$ and $\lambda_{S,1}^{(j)} > 0$. If $\lambda_{X,1}^{(k)} > 0$, then the inequality (13) holds at least for d_k sufficiently close to $d_{\min}^{(k)}$; if $\lambda_{X,1}^{(k)} = 0$, which implies $k = j = \ell$, then the inequality (13) always holds. The inequality (14) holds at least for d_k sufficiently close to $d_{\min}^{(k)}$ unless $\lambda_{X,2} = 0$ and $\lambda_{S,2} > \lambda_{S,1}^{(j)}$.

Example 2: Let $\gamma_X = 1$, $\rho_X = 0.8$, $\gamma_Z = 0.5$, $\rho_Z = 0.1$, and $k = 2$. In this case $d_{\min}^{(k)} \approx 0.2799$. It can be verified that $\rho_S \approx 0.5667 > 0$ and $(k-1)\lambda_{X,2}^2(\lambda_{S,1}^{(k)})^2 = 0.2209 < 4k(\lambda_{X,1}^{(k)})^2\lambda_{S,2}^2 = 10.9512$. In light of Remark 3, condition i) in Theorem 2 is satisfied. In Fig. 2 we plot $d_j^{(k)}(d_k)$ with $d_k \in (d_{\min}^{(k)}, \gamma_X)$ for various j .

III. PROOF OF PART 1) OF THEOREM 2

The following lemma can be obtained by adapting the classical result by Berger [40] and Tung [41] to the current setting.

Lemma 1: For any auxiliary random vector $V \triangleq (V_1, \dots, V_\ell)^T$ jointly distributed with (X, Z, S) such that $\{X, Z, (S_{i'})_{i' \in \{1, \dots, \ell\} \setminus \{i\}}, (V_{i'})_{i' \in \{1, \dots, \ell\} \setminus \{i\}}\} \leftrightarrow S_i \leftrightarrow V_i$ form a Markov chain, $i = 1, \dots, \ell$, and any (r, d_k, \dots, d_ℓ) such that

$$r1_k \in \mathcal{R}(\mathcal{A}), \quad \mathcal{A} \subseteq \{1, \dots, \ell\} \quad \text{with } |\mathcal{A}| = k,$$

$$d_{|\mathcal{A}|} \geq \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} \mathbb{E}[(X_i - \mathbb{E}[X_i | (V_{i'})_{i' \in \mathcal{A}}])^2],$$

$$\mathcal{A} \subseteq \{1, \dots, \ell\} \quad \text{with } |\mathcal{A}| \geq k,$$

where $\mathcal{R}(\mathcal{A})$ denotes the set of $(r_i)_{i \in \mathcal{A}}$ satisfying

$$\sum_{i \in \mathcal{B}} r_i \geq I((S_i)_{i \in \mathcal{B}}; (V_i)_{i \in \mathcal{B}} | (V_i)_{i \in \mathcal{A} \setminus \mathcal{B}}), \quad \emptyset \subset \mathcal{B} \subseteq \mathcal{A},$$

we have

$$(r, d_k, \dots, d_\ell) \in \mathcal{RD}_k.$$

Remark 5: Roughly speaking, $r1_k \in \mathcal{R}(\mathcal{A})$ is the constraint induced by the conventional Berger-Tung inner bound, which ensures the joint decodability of $(V_i)_{i \in \mathcal{A}}$ based on the outputs

of encoders specified by \mathcal{A} . Therefore, if $r1_k \in \mathcal{R}(\mathcal{A})$ holds for every $\mathcal{A} \subseteq \{1, \dots, \ell\}$ with $|\mathcal{A}| = k$, then the outputs of any k or more encoders are jointly decodable and the corresponding V_i 's can be leveraged to reconstruct the target signals via MMSE estimation.

Equipped with this lemma, we are in a position to prove Part 1) of Theorem 2. Let $Q \triangleq (Q_1, \dots, Q_\ell)^T$ be an ℓ -dimensional zero-mean Gaussian random vector with covariance matrix

$$\Lambda_Q \triangleq \text{diag}^{(\ell)}(\lambda_Q, \dots, \lambda_Q) \succ 0.$$

Moreover, we assume that Q is independent of (X, Z, S) , and let

$$V_i \triangleq S_i + Q_i, \quad i = 1, \dots, \ell.$$

Clearly, $V \triangleq (V_1, \dots, V_\ell)^T$ satisfies the condition specified in Lemma 1. Let

$$r \triangleq \frac{1}{k} I(S_1, \dots, S_k; V_1, \dots, V_k),$$

$$d_j \triangleq \frac{1}{j} \sum_{i=1}^j \mathbb{E}[(X_i - \mathbb{E}[X_i | V_1, \dots, V_j])^2], \quad j = k, \dots, \ell.$$

It is easy to show that $r1_k \in \mathcal{R}(\mathcal{A})$ for all $\mathcal{A} \subseteq \{1, \dots, \ell\}$ with $|\mathcal{A}| = k$ by leveraging the contra-polymatroid structure [42] of $\mathcal{R}(\mathcal{A})$ and the symmetry of the underlying distributions. Let $\Lambda_Q^{(j)}$ denote the leading $j \times j$ principal submatrix of Λ_Q , $j = k, \dots, \ell$. We have

$$\begin{aligned} r &= \frac{1}{k} (h(V_1, \dots, V_k) - h(V_1, \dots, V_k | S_1, \dots, S_k)) \\ &= \frac{1}{k} (h(S_1 + Q_1, \dots, S_k + Q_k) - h(Q_1, \dots, Q_k)) \\ &= \frac{1}{2k} \log \frac{\det(\Gamma_S^{(k)} + \Lambda_Q^{(k)})}{\det(\Lambda_Q^{(k)})} \\ &= \frac{1}{2k} \log \frac{\det(\Lambda_S^{(k)} + \Lambda_Q^{(k)})}{\det(\Lambda_Q^{(k)})} \\ &= \frac{1}{2k} \log \frac{(\lambda_{S,1}^{(k)} + \lambda_Q)(\lambda_{S,2} + \lambda_Q)^{k-1}}{\lambda_Q^k}. \end{aligned}$$

Moreover, for $j = k, \dots, \ell$,

$$\begin{aligned} d_j &= \frac{1}{j} \text{tr}(\Gamma_X^{(j)} - \Gamma_X^{(j)}(\Gamma_S^{(j)} + \Lambda_Q^{(j)})^{-1} \Gamma_X^{(j)}) \\ &= \frac{1}{j} \text{tr}(\Lambda_X^{(j)} - \Lambda_X^{(j)}(\Lambda_S^{(j)} + \Lambda_Q^{(j)})^{-1} \Lambda_X^{(j)}) \\ &= \frac{\lambda_{X,1}^{(j)}(\lambda_{Z,1}^{(j)} + \lambda_Q)}{j(\lambda_{S,1}^{(j)} + \lambda_Q)} + \frac{(j-1)\lambda_{X,2}(\lambda_{Z,2} + \lambda_Q)}{j(\lambda_{S,2} + \lambda_Q)}, \end{aligned}$$

which is a strictly increasing function of λ_Q , converging to $d_{\min}^{(j)}$ as $\lambda_Q \rightarrow 0$ and to γ_X as $\lambda_Q \rightarrow \infty$. One can readily complete the proof of Part 1) of Theorem 2 by invoking Lemma 1.

IV. PROOF OF PART 2) AND PART 3) OF THEOREM 2

Fix k and j with $1 \leq k \leq j \leq \ell$. We proceed by considering the nondegenerate case $\Gamma_S^{(j)} \succ 0$ and the degenerate case $\Gamma_S^{(j)} = 0$ separately. The argument for the nondegenerate case is quite sophisticated and is roughly divided into 3 steps:

- 1) augment the probability space via a fictitious signal-noise decomposition;
- 2) establish certain constraints on the relevant parameters;
- 3) analyze the resulting convex optimization problems.

In contrast, the degenerate case can be easily handled due to its connection with the multiple description problem and the centralized remote source coding problem.

A. Nondegenerate Case

Step 1: Let $(S_1, \dots, S_j)^T = (U_1, \dots, U_j)^T + (W_1, \dots, W_j)^T$ be a fictitious signal-noise decomposition of $(S_1, \dots, S_j)^T$, where $(U_1, \dots, U_j)^T$ and $(W_1, \dots, W_j)^T$ are two mutually independent j -dimensional zero-mean Gaussian vectors with covariance matrices

$$\Gamma_U^{(j)} \succ 0,$$

$$\Lambda_W^{(j)} \triangleq \text{diag}^{(j)}(\lambda_W, \dots, \lambda_W) \succ 0,$$

respectively. We then construct the auxiliary random processes $\{(U_1(t), \dots, U_j(t))^T\}_{t=1}^\infty$ and $\{(W_1(t), \dots, W_\ell(t))^T\}_{t=1}^\infty$ accordingly.

It is worth mentioning that the idea of augmenting the probability space via the introduction of auxiliary random processes is inspired by [8], [10], [13]–[15], [18], [24], [26], and [28]. Our construction (without the symmetry constraint) can be viewed as a generalization of that in [10], which is restricted to the special case where the corruptive noises are absent. It should also be contrasted with the conventional approach where $(U_1, \dots, U_j)^T$ and $(W_1, \dots, W_j)^T$ are set respectively to be $(X_1, \dots, X_j)^T$ and $(Z_1, \dots, Z_j)^T$ (with the components of $(Z_1, \dots, Z_j)^T$ assumed to be mutually independent); our construction is more flexible and often yields stronger results. The fictitious signal-noise decomposition is closely related to the Markov coupling argument in [43]. One subtle difference is that the fictitious decomposition is specified for $(S_1, \dots, S_j)^T$ instead of $(S_1, \dots, S_\ell)^T$. As a consequence, we can choose λ_W from $(0, \min\{\lambda_{S,1}^{(j)}, \lambda_{S,2}\})$, which may offer more freedom than $(0, \min\{\lambda_{S,1}^{(\ell)}, \lambda_{S,2}\})$ since $\min\{\lambda_{S,1}^{(j)}, \lambda_{S,2}\} \geq \min\{\lambda_{S,1}^{(\ell)}, \lambda_{S,2}\}$ and the inequality is strict when $\rho_S < 0$ and $j < \ell$.

Step 2: In view of Definition 1, for any $(r, d_k, \dots, d_\ell) \in \mathcal{RD}_k$ and $\epsilon > 0$, there exist encoding functions $\phi_i^{(n)} : \mathbb{R}^n \rightarrow \mathcal{C}_i^{(n)}$, $i = 1, \dots, j$, such that

$$\frac{1}{kn} \sum_{i \in \mathcal{A}} \log |\mathcal{C}_i^{(n)}| \leq r + \epsilon,$$

$$\mathcal{A} \subseteq \{1, \dots, j\} \quad \text{with } |\mathcal{A}| = k, \quad (15)$$

$$\frac{1}{kn} \sum_{i \in \mathcal{A}} \sum_{t=1}^n \mathbb{E}[(X_i(t) - \hat{X}_{i,\mathcal{A}}(t))^2] \leq d_k + \epsilon,$$

$$\mathcal{A} \subseteq \{1, \dots, j\} \quad \text{with } |\mathcal{A}| = k, \quad (16)$$

$$\frac{1}{jn} \sum_{i=1}^j \sum_{t=1}^n \mathbb{E}[(X_i(t) - \hat{X}_{i,\{1, \dots, j\}}(t))^2] \leq d_j + \epsilon.$$

We have

$$\begin{aligned}
& \sum_{i \in \mathcal{A}} \log |\mathcal{C}_i^{(n)}| \\
& \geq H((\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& = I((U_i^n)_{i \in \mathcal{A}}; (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& \quad + H((\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}} | (U_i^n)_{i \in \mathcal{A}}) \\
& = I((U_i^n)_{i \in \mathcal{A}}; (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& \quad + I((S_i^n)_{i \in \mathcal{A}}; (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}} | (U_i^n)_{i \in \mathcal{A}}) \\
& = h((U_i^n)_{i \in \mathcal{A}}) + h((W_i^n)_{i \in \mathcal{A}}) \\
& \quad - h((U_i^n)_{i \in \mathcal{A}} | (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& \quad - h((S_i^n)_{i \in \mathcal{A}} | (U_i^n)_{i \in \mathcal{A}}, (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& = \frac{n}{2} \log((2\pi e)^k \det(\Gamma_U^{(k)})) + \frac{n}{2} \log((2\pi e)^k \det(\Lambda_W^{(k)})) \\
& \quad - h((U_i^n)_{i \in \mathcal{A}} | (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& \quad - h((S_i^n)_{i \in \mathcal{A}} | (U_i^n)_{i \in \mathcal{A}}, (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}), \quad (17)
\end{aligned}$$

where $\Gamma_U^{(k)}$ and $\Lambda_W^{(k)}$ denote the leading $k \times k$ principal submatrices of $\Gamma_U^{(j)}$ and $\Lambda_W^{(j)}$, respectively. For $t = 1, \dots, n$, let

$$\begin{aligned}
\Sigma_{\mathcal{A}}(t) & \triangleq \mathbb{E}[(U_i(t) - \hat{U}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T (U_i(t) - \hat{U}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}], \\
\Delta_{\mathcal{A}}(t) & \triangleq \mathbb{E}[(S_i(t) - \tilde{S}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T (S_i(t) - \tilde{S}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}],
\end{aligned}$$

where

$$\begin{aligned}
\hat{U}_{i,\mathcal{A}}(t) & \triangleq \mathbb{E}[U_i(t) | (\phi_{i'}^{(n)}(S_{i'}^n))_{i' \in \mathcal{A}}], \quad i \in \mathcal{A}, \\
\tilde{S}_{i,\mathcal{A}}(t) & \triangleq \mathbb{E}[S_i(t) | (U_{i'}^n)_{i' \in \mathcal{A}}, (\phi_{i'}^{(n)}(S_{i'}^n))_{i' \in \mathcal{A}}], \quad i \in \mathcal{A}.
\end{aligned}$$

Moreover, let

$$\begin{aligned}
\Sigma_{\mathcal{A}} & \triangleq \frac{1}{n} \sum_{t=1}^n \Sigma_{\mathcal{A}}(t), \\
\Delta_{\mathcal{A}} & \triangleq \frac{1}{n} \sum_{t=1}^n \Delta_{\mathcal{A}}(t).
\end{aligned}$$

It can be verified that

$$\begin{aligned}
& h((U_i^n)_{i \in \mathcal{A}} | (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& = \sum_{t=1}^n h((U_i(t))_{i \in \mathcal{A}} | (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}, (U_i^{t-1})_{i \in \mathcal{A}}) \\
& \leq \sum_{t=1}^n h((U_i(t))_{i \in \mathcal{A}} | (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& = \sum_{t=1}^n h((U_i(t) - \hat{U}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}} | (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& \leq \sum_{t=1}^n h((U_i(t) - \hat{U}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}) \\
& \leq \sum_{t=1}^n \frac{1}{2} \log((2\pi e)^k \det(\Sigma_{\mathcal{A}}(t))) \quad (18) \\
& \leq \frac{n}{2} \log((2\pi e)^k \det(\Sigma_{\mathcal{A}})), \quad (19)
\end{aligned}$$

where (18) is due to the maximum differential entropy lemma [44, p. 21], and (19) is due to the concavity of the

log-determinant function. Similarly, we have

$$\begin{aligned}
& h((S_i^n)_{i \in \mathcal{A}} | (U_i^n)_{i \in \mathcal{A}}, (\phi_i^{(n)}(S_i^n))_{i \in \mathcal{A}}) \\
& \leq \frac{n}{2} \log((2\pi e)^k \det(\Delta_{\mathcal{A}})). \quad (20)
\end{aligned}$$

Combining (15), (17), (19), and (20) gives

$$\frac{1}{2k} \log \frac{\det(\Gamma_U^{(k)}) \det(\Lambda_W^{(k)})}{\det(\Sigma_{\mathcal{A}}) \det(\Delta_{\mathcal{A}})} \leq r + \epsilon. \quad (21)$$

For $t = 1, \dots, n$, let

$$D_{\mathcal{A}}(t) \triangleq \mathbb{E}[(S_i(t) - \hat{S}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T (S_i(t) - \hat{S}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}],$$

where

$$\hat{S}_{i,\mathcal{A}}(t) \triangleq \mathbb{E}[S_i(t) | (\phi_{i'}^{(n)}(S_{i'}^n))_{i' \in \mathcal{A}}], \quad i \in \mathcal{A}.$$

Moreover, let

$$D_{\mathcal{A}} \triangleq \frac{1}{n} \sum_{t=1}^n D_{\mathcal{A}}(t).$$

Clearly, we have

$$0 \prec D_{\mathcal{A}} \preceq \Gamma_S^{(k)}. \quad (22)$$

Furthermore, as shown in Appendix B,

$$\begin{aligned}
\Sigma_{\mathcal{A}} & = \Gamma_U^{(k)} (\Gamma_S^{(k)})^{-1} D_{\mathcal{A}} (\Gamma_S^{(k)})^{-1} \Gamma_U^{(k)} + \Gamma_U^{(k)} \\
& \quad - \Gamma_U^{(k)} (\Gamma_S^{(k)})^{-1} \Gamma_U^{(k)}, \quad (23)
\end{aligned}$$

$$\Delta_{\mathcal{A}} \preceq (D_{\mathcal{A}}^{-1} + (\Lambda_W^{(k)})^{-1} - (\Gamma_S^{(k)})^{-1})^{-1}. \quad (24)$$

The argument for (23) can also be leveraged to prove

$$\begin{aligned}
& \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{t=1}^n \mathbb{E}[(X_i(t) - \hat{X}_{i,\mathcal{A}}(t))^2] \\
& = \text{tr}(\Gamma_X^{(k)} (\Gamma_S^{(k)})^{-1} D_{\mathcal{A}} (\Gamma_S^{(k)})^{-1} \Gamma_X^{(k)} + \Gamma_X^{(k)} \\
& \quad - \Gamma_X^{(k)} (\Gamma_S^{(k)})^{-1} \Gamma_X^{(k)}),
\end{aligned}$$

which, together with (16), implies

$$\begin{aligned}
& \text{tr}(\Gamma_X^{(k)} (\Gamma_S^{(k)})^{-1} D_{\mathcal{A}} (\Gamma_S^{(k)})^{-1} \Gamma_X^{(k)} + \Gamma_X^{(k)} \\
& \quad - \Gamma_X^{(k)} (\Gamma_S^{(k)})^{-1} \Gamma_X^{(k)}) \leq k(d_k + \epsilon). \quad (25)
\end{aligned}$$

For $t = 1, \dots, n$, let

$$\begin{aligned}
& \Delta_{\{1, \dots, j\}}(t) \\
& \triangleq \mathbb{E}[(S_1(t) - \tilde{S}_{1,\{1, \dots, j\}}(t), \dots, S_j(t) - \tilde{S}_{j,\{1, \dots, j\}}(t))^T \\
& \quad (S_1(t) - \tilde{S}_{1,\{1, \dots, j\}}(t), \dots, S_j(t) - \tilde{S}_{j,\{1, \dots, j\}}(t))], \\
& D_{\{1, \dots, j\}}(t) \\
& \triangleq \mathbb{E}[(S_1(t) - \hat{S}_{1,\{1, \dots, j\}}(t), \dots, S_j(t) - \hat{S}_{j,\{1, \dots, j\}}(t))^T \\
& \quad (S_1(t) - \hat{S}_{1,\{1, \dots, j\}}(t), \dots, S_j(t) - \hat{S}_{j,\{1, \dots, j\}}(t))], \\
& \delta_i(t) \triangleq \mathbb{E}[(S_i(t) - \tilde{S}_i(t))^2], \quad i = 1, \dots, j,
\end{aligned}$$

where

$$\begin{aligned}
& \tilde{S}_{i,\{1, \dots, j\}}(t) \\
& \triangleq \mathbb{E}[S_i(t) | U_1^n, \dots, U_j^n, \phi_1^{(n)}(S_1^n), \dots, \phi_j^{(n)}(S_j^n)], \\
& \quad i = 1, \dots, j,
\end{aligned}$$

$$\begin{aligned}\hat{S}_{i,\{1,\dots,j\}}(t) &\triangleq \mathbb{E}[S_i(t)|\phi_1^{(n)}(S_1^n), \dots, \phi_j^{(n)}(S_j^n)], \\ &\quad i = 1, \dots, j, \\ \tilde{S}_i(t) &\triangleq \mathbb{E}[S_i(t)|U_i^n, \phi_i^{(n)}(S_i^n)], \quad i = 1, \dots, j.\end{aligned}$$

Moreover, let

$$\begin{aligned}\Delta_{\{1,\dots,j\}} &\triangleq \frac{1}{n} \sum_{t=1}^n \Delta_{\{1,\dots,j\}}(t), \\ D_{\{1,\dots,j\}} &\triangleq \frac{1}{n} \sum_{t=1}^n D_{\{1,\dots,j\}}(t), \\ \delta_i &\triangleq \sum_{t=1}^n \delta_i(t), \quad i = 1, \dots, j.\end{aligned}$$

The argument for (24) and (25) can be leveraged to show that

$$\Delta_{\{1,\dots,j\}} \preceq (D_{\{1,\dots,j\}}^{-1} + (\Lambda_W^{(j)})^{-1} - (\Gamma_S^{(j)})^{-1})^{-1}, \quad (26)$$

$$\begin{aligned}\text{tr}(\Gamma_X^{(j)}(\Gamma_S^{(j)})^{-1}D_{\{1,\dots,j\}}(\Gamma_S^{(j)})^{-1}\Gamma_X^{(j)} + \Gamma_X^{(j)} \\ - \Gamma_X^{(j)}(\Gamma_S^{(j)})^{-1}\Gamma_X^{(j)}) \leq j(d_j + \epsilon).\end{aligned} \quad (27)$$

It is also clear that

$$0 < \delta_i, \quad i = 1, \dots, \ell. \quad (28)$$

Furthermore, in view of the fact that $S_i^n = U_i^n + W_i^n$, $i = 1, \dots, j$, and that $(U_1^n, \dots, U_j^n), W_1^n, \dots, W_j^n$ are mutually independent, we must have

$$\Delta_{\mathcal{A}} = \text{diag}^{(k)}(\delta_i)_{i \in \mathcal{A}}, \quad (29)$$

$$\Delta_{\{1,\dots,j\}} = \text{diag}^{(j)}(\delta_1, \dots, \delta_j). \quad (30)$$

Combining (21)–(30), sending $\epsilon \rightarrow 0$, and invoking a symmetrization and convexity argument² shows that there exist $D^{(k)}$, $D^{(j)}$, and δ satisfying the following set of inequalities

$$\frac{1}{2k} \log \frac{\det(\Gamma_U^{(k)})}{\det(\Sigma^{(k)})} + \frac{1}{2} \log \frac{\lambda_W}{\delta} \leq r, \quad (31)$$

$$0 \prec D^{(k)} \preceq \Gamma_S^{(k)}, \quad (32)$$

$$0 < \delta, \quad (33)$$

$$\begin{aligned}\text{diag}^{(k)}(\delta, \dots, \delta) \\ \preceq ((D^{(k)})^{-1} + (\Lambda_W^{(k)})^{-1} - (\Gamma_S^{(k)})^{-1})^{-1},\end{aligned} \quad (34)$$

$$\begin{aligned}\text{tr}(\Gamma_X^{(k)}(\Gamma_S^{(k)})^{-1}D^{(k)}(\Gamma_S^{(k)})^{-1}\Gamma_X^{(k)} + \Gamma_X^{(k)} \\ - \Gamma_X^{(k)}(\Gamma_S^{(k)})^{-1}\Gamma_X^{(k)}) \leq kd_k,\end{aligned} \quad (35)$$

$$\begin{aligned}\text{diag}^{(j)}(\delta, \dots, \delta) \\ \preceq ((D^{(j)})^{-1} + (\Lambda_W^{(j)})^{-1} - (\Gamma_S^{(j)})^{-1})^{-1},\end{aligned} \quad (36)$$

$$\begin{aligned}\text{tr}(\Gamma_X^{(j)}(\Gamma_S^{(j)})^{-1}D^{(j)}(\Gamma_S^{(j)})^{-1}\Gamma_X^{(j)} + \Gamma_X^{(j)} \\ - \Gamma_X^{(j)}(\Gamma_S^{(j)})^{-1}\Gamma_X^{(j)}) \leq jd_j,\end{aligned} \quad (37)$$

²It can be verified that that $D_{\mathcal{A}}$, $D_{\{1,\dots,j\}}$, $\Delta_{\mathcal{A}}$, and $\Delta_{\{1,\dots,j\}}$ can be replaced simultaneously by their respective symmetrized counterparts $\text{sym}(\frac{1}{k} \sum_{\mathcal{A} \subseteq \{1,\dots,j\}; |\mathcal{A}|=k} D_{\mathcal{A}})$, $\text{sym}(D_{\{1,\dots,j\}})$, $\text{sym}(\frac{1}{j} \sum_{\mathcal{A} \subseteq \{1,\dots,j\}; |\mathcal{A}|=k} \Delta_{\mathcal{A}})$, and $\text{sym}(\Delta_{\{1,\dots,j\}})$ without violating the relevant inequalities. Here $\text{sym}(\cdot)$ maps the given matrix to a matrix of the same size with all its diagonal (resp. off-diagonal) entries equal to the mean of the diagonal (resp. off-diagonal) entries of the given matrix.

where

$$\begin{aligned}D^{(k)} &= \Theta^{(k)} \text{diag}^{(k)}(d_1^{(k)}, d_2^{(k)}, \dots, d_2^{(k)})(\Theta^{(k)})^T, \\ D^{(j)} &= \Theta^{(j)} \text{diag}^{(j)}(d_1^{(j)}, d_2^{(j)}, \dots, d_2^{(j)})(\Theta^{(j)})^T\end{aligned}$$

for some $d_1^{(k)}, d_2^{(k)}, d_1^{(j)}, d_2^{(j)}$, and

$$\Sigma^{(k)} \triangleq \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}D^{(k)}(\Gamma_S^{(k)})^{-1}\Gamma_U^{(k)} + \Gamma_U^{(k)} - \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}\Gamma_U^{(k)}.$$

Equivalently, (31)–(37) can be written as

$$\begin{aligned}\frac{1}{2k} \log \frac{(\lambda_{S,1}^{(k)})^2}{(\lambda_{S,1}^{(k)} - \lambda_W)d_1^{(k)} + \lambda_{S,1}^{(k)}\lambda_W} \\ + \frac{k-1}{2k} \log \frac{\lambda_{S,2}^2}{(\lambda_{S,2} - \lambda_W)d_2^{(k)} + \lambda_{S,2}\lambda_W} + \frac{1}{2} \log \frac{\lambda_W}{\delta} \\ \leq r,\end{aligned} \quad (38)$$

$$0 < d_1^{(k)} \leq \lambda_{S,1}^{(k)}, \quad (39)$$

$$0 < d_2^{(k)} \leq \lambda_{S,2}, \quad (40)$$

$$0 < \delta, \quad (41)$$

$$\delta \leq ((d_1^{(k)})^{-1} + \lambda_W^{-1} - (\lambda_{S,1}^{(k)})^{-1})^{-1}, \quad (42)$$

$$\delta \leq ((d_2^{(k)})^{-1} + \lambda_W^{-1} - \lambda_{S,2}^{-1})^{-1}, \quad (43)$$

$$\begin{aligned}(\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2}d_1^{(k)} + \lambda_{X,1}^{(k)} - (\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-1} \\ + (k-1)(\lambda_{X,2}^2\lambda_{S,2}^{-2}d_2^{(k)} + \lambda_{X,2} - \lambda_{X,2}^2\lambda_{S,2}^{-1}) \\ \leq kd_k,\end{aligned} \quad (44)$$

$$\delta \leq ((d_1^{(j)})^{-1} + \lambda_W^{-1} - (\lambda_{S,1}^{(j)})^{-1})^{-1}, \quad (45)$$

$$\delta \leq ((d_2^{(j)})^{-1} + \lambda_W^{-1} - \lambda_{S,2}^{-1})^{-1}, \quad (46)$$

$$\begin{aligned}(\lambda_{X,1}^{(j)})^2(\lambda_{S,1}^{(j)})^{-2}d_1^{(j)} + \lambda_{X,1}^{(j)} - (\lambda_{X,1}^{(j)})^2(\lambda_{S,1}^{(j)})^{-1} \\ + (j-1)(\lambda_{X,2}^2\lambda_{S,2}^{-2}d_2^{(j)} + \lambda_{X,2} - \lambda_{X,2}^2\lambda_{S,2}^{-1}) \\ \leq jd_j.\end{aligned} \quad (47)$$

Step 3: Subcase 1: $\lambda_{S,1}^{(j)} \geq \lambda_{S,2} > 0$. We can send $\lambda_W \rightarrow \lambda_{S,2}$ and deduce from (38), (42), (43), (45), and (46) that

$$\eta(d_1^{(k)}, d_2^{(k)}, \delta) \leq r, \quad (48)$$

$$\delta \leq ((d_1^{(k)})^{-1} + \lambda_{S,2}^{-1} - (\lambda_{S,1}^{(k)})^{-1})^{-1}, \quad (49)$$

$$\delta \leq d_2^{(k)}, \quad (50)$$

$$\delta \leq ((d_1^{(j)})^{-1} + \lambda_{S,2}^{-1} - (\lambda_{S,1}^{(j)})^{-1})^{-1}, \quad (51)$$

$$\delta \leq d_2^{(j)}, \quad (52)$$

where

$$\begin{aligned}\eta(d_1^{(k)}, d_2^{(k)}, \delta) \\ \triangleq \frac{1}{2k} \log \frac{(\lambda_{S,1}^{(k)})^2}{(\lambda_{S,1}^{(k)} - \lambda_{S,2})d_1^{(k)} + \lambda_{S,1}^{(k)}\lambda_{S,2}} + \frac{1}{2} \log \frac{\lambda_{S,2}}{\delta}.\end{aligned}$$

Furthermore, combining (47), (51), and (52) gives

$$\begin{aligned}d_j \geq \frac{1}{j}((\lambda_{X,1}^{(j)})^2(\lambda_{S,1}^{(j)})^{-2}(\delta^{-1} + (\lambda_{S,1}^{(j)})^{-1} - \lambda_{S,2}^{-1})^{-1} \\ + \lambda_{X,1}^{(j)} - (\lambda_{X,1}^{(j)})^2(\lambda_{S,1}^{(j)})^{-1}) \\ + \frac{j-1}{j}(\lambda_{X,2}^2\lambda_{S,2}^{-2}\delta + \lambda_{X,2} - \lambda_{X,2}^2\lambda_{S,2}^{-1}).\end{aligned} \quad (53)$$

Now consider the following convex optimization problem:

$$\min_{d_1^{(k)}, d_2^{(k)}, \delta} \eta(d_1^{(k)}, d_2^{(k)}, \delta) \quad (\mathbf{P})$$

subject to (39), (40), (41), (49), (50), and (44). According to the Karush-Kuhn-Tucker conditions, $(d_1^{(k)}, d_2^{(k)}, \delta)$ is a minimizer of the convex optimization problem (\mathbf{P}) if and only if (39), (40), (41), (49), (50), and (44) are satisfied, and there exist nonnegative a_1, a_2, b_1, b_2, c such that

$$\begin{aligned} & \frac{\lambda_{S,2} - \lambda_{S,1}^{(k)}}{2k((\lambda_{S,1}^{(k)} - \lambda_{S,2}^{(k)})d_1^{(k)} + \lambda_{S,1}^{(k)}\lambda_{S,2})} + a_1 \\ & - b_1(1 + \lambda_{S,2}^{-1}d_1^{(k)} - (\lambda_{S,1}^{(k)})^{-1}d_1^{(k)})^{-2} \\ & + c(\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2} = 0, \end{aligned} \quad (54)$$

$$a_2 - b_2 + c(k-1)\lambda_{X,2}^2\lambda_{S,2}^{-2} = 0, \quad (55)$$

$$-\frac{1}{2\delta} + b_1 + b_2 = 0, \quad (56)$$

$$a_1(d_1^{(k)} - \lambda_{S,1}^{(k)}) = 0, \quad (57)$$

$$a_2(d_2^{(k)} - \lambda_{S,2}) = 0, \quad (58)$$

$$b_1(\delta - ((d_1^{(k)})^{-1} + \lambda_{S,2}^{-1} - (\lambda_{S,1}^{(k)})^{-1})^{-1}) = 0, \quad (59)$$

$$b_2(\delta - d_2^{(k)}) = 0, \quad (60)$$

$$\begin{aligned} & c((\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2}d_1^{(k)} + \lambda_{X,1}^{(k)} - (\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-1} \\ & + (k-1)(\lambda_{X,2}^2\lambda_{S,2}^{-2}d_2^{(k)} + \lambda_{X,2} - \lambda_{X,2}^2\lambda_{S,2}^{-1}) - kd_k) = 0. \end{aligned} \quad (61)$$

Assume $d_k \in (d_{\min}^{(k)}, \gamma_X)$. It can be verified via algebraic manipulations that $\eta(d_1^{(k)}, d_2^{(k)}, \delta) = \bar{r}(d_k)$ for

$$\begin{aligned} d_1^{(k)} & \triangleq ((\lambda_{S,1}^{(k)})^{-1} + (\lambda_Q^{(k)})^{-1})^{-1}, \\ d_2^{(k)} & \triangleq (\lambda_{S,2}^{-1} + (\lambda_Q^{(k)})^{-1})^{-1}, \\ \delta & \triangleq (\lambda_{S,2}^{-1} + (\lambda_Q^{(k)})^{-1})^{-1}, \end{aligned} \quad (62)$$

where $\lambda_Q^{(k)}$ is given by (4). We shall identify the condition under which this specific $(d_1^{(k)}, d_2^{(k)}, \delta)$ is a minimizer of (\mathbf{P}) . Clearly, (59)–(61) are satisfied. Moreover, in view of (57), (58) as well as the fact that $d_1^{(k)} < \lambda_{S,1}^{(k)}$ and $d_2^{(k)} < \lambda_{S,2}$, we must have

$$a_m = 0, \quad m = 1, 2,$$

which, together with (54)–(56), implies

$$\begin{aligned} b_1 & = \frac{d_2^{(k)} - d_1^{(k)} + 2kc(\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2}(d_1^{(k)})^2}{2k(d_2^{(k)})^2}, \\ b_2 & = (k-1)c\lambda_{X,2}^2\lambda_{S,2}^{-2}, \\ c & = \frac{d_1^{(k)} + (k-1)d_2^{(k)}}{(\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2}(d_1^{(k)})^2 + (k-1)\lambda_{X,2}^2\lambda_{S,2}^{-2}(d_2^{(k)})^2} \\ & \quad \times \frac{1}{2k}. \end{aligned}$$

It is obvious that b_2 and c are nonnegative. Therefore, it suffices to have $b_1 \geq 0$, which is equivalent to condition (11). Moreover, under this condition, every minimizer $(d_1^{(k)}, d_2^{(k)}, \delta)$ of (\mathbf{P}) must satisfy (62) due to the fact that $\frac{1}{2} \log \frac{\lambda_{S,2}}{\delta}$ is

a strictly convex function of δ (in other words, (48), (39), (40), (41), (49), (50), and (44) imply that δ is uniquely determined and is given by (62) when $r = \bar{r}(d_k)$). Hence, under condition (11), when $r = \bar{r}(d_k)$, we can deduce $d_j \geq d_j^{(k)}(d_k)$ by substituting (62) into (53).

Subcase 2: $\lambda_{S,2} \geq \lambda_{S,1}^{(j)} > 0$. We can send $\lambda_W \rightarrow \lambda_{S,1}^{(j)}$ and deduce from (38), (42), (43), (45), and (46) that

$$\hat{\eta}(d_1^{(k)}, d_2^{(k)}, \delta) \leq r, \quad (63)$$

$$\delta \leq ((d_1^{(k)})^{-1} + (\lambda_{S,1}^{(j)})^{-1} - (\lambda_{S,1}^{(k)})^{-1})^{-1}, \quad (64)$$

$$\delta \leq ((d_2^{(k)})^{-1} + (\lambda_{S,1}^{(j)})^{-1} - \lambda_{S,2}^{-1})^{-1}, \quad (65)$$

$$\delta \leq d_1^{(j)}, \quad (66)$$

$$\delta \leq ((d_2^{(j)})^{-1} + (\lambda_{S,1}^{(j)})^{-1} - \lambda_{S,2}^{-1})^{-1}, \quad (67)$$

where

$$\begin{aligned} \hat{\eta}(d_1^{(k)}, d_2^{(k)}, \delta) & \triangleq \frac{1}{2k} \log \frac{(\lambda_{S,1}^{(k)})^2}{(\lambda_{S,1}^{(k)} - \lambda_{S,1}^{(j)})d_1^{(k)} + \lambda_{S,1}^{(k)}\lambda_{S,1}^{(j)}} \\ & + \frac{k-1}{2k} \log \frac{\lambda_{S,2}^2}{(\lambda_{S,2} - \lambda_{S,1}^{(j)})d_2^{(k)} + \lambda_{S,2}\lambda_{S,1}^{(j)}} + \frac{1}{2} \log \frac{\lambda_{S,1}^{(j)}}{\delta}. \end{aligned}$$

Furthermore, combining (47), (66), and (67) gives

$$\begin{aligned} d_j & \geq \frac{1}{j}((\lambda_{X,1}^{(j)})^2(\lambda_{S,1}^{(j)})^{-2}\delta + \lambda_{X,1}^{(j)} - (\lambda_{X,1}^{(j)})^2(\lambda_{S,1}^{(j)})^{-1}) \\ & + \frac{j-1}{j}(\lambda_{X,2}^2\lambda_{S,2}^{-2}(\delta^{-1} + \lambda_{S,2}^{-1} - (\lambda_{S,1}^{(j)})^{-1})^{-1} \\ & + \lambda_{X,2} - \lambda_{X,2}^2\lambda_{S,2}^{-1}). \end{aligned} \quad (68)$$

Now consider the following convex optimization problem:

$$\min_{d_1^{(k)}, d_2^{(k)}, \delta} \hat{\eta}(d_1^{(k)}, d_2^{(k)}, \delta) \quad (\hat{\mathbf{P}})$$

subject to (39), (40), (41), (64), (65), and (44). According to the Karush-Kuhn-Tucker conditions, $(d_1^{(k)}, d_2^{(k)}, \delta)$ is a minimizer of the convex optimization problem $(\hat{\mathbf{P}})$ if and only if (39), (40), (41), (64), (65), and (44) are satisfied, and there exist nonnegative $\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2, \hat{c}$ such that

$$\begin{aligned} & \frac{\lambda_{S,1}^{(j)} - \lambda_{S,1}^{(k)}}{2k((\lambda_{S,1}^{(k)} - \lambda_{S,1}^{(j)})d_1^{(k)} + \lambda_{S,1}^{(k)}\lambda_{S,1}^{(j)})} + \hat{a}_1 \\ & - \hat{b}_1(1 + (\lambda_{S,1}^{(j)})^{-1}d_1^{(k)} - (\lambda_{S,1}^{(k)})^{-1}d_1^{(k)})^{-2} \\ & + \hat{c}(\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2} = 0, \end{aligned} \quad (69)$$

$$\begin{aligned} & \frac{(k-1)(\lambda_{S,1}^{(j)} - \lambda_{S,2})}{2k((\lambda_{S,2} - \lambda_{S,1}^{(j)})d_2^{(k)} + \lambda_{S,2}\lambda_{S,1}^{(j)})} + \hat{a}_2 \\ & - \hat{b}_2(1 + (\lambda_{S,1}^{(j)})^{-1}d_2^{(k)} - \lambda_{S,2}^{-1}d_2^{(k)})^{-2} \\ & + \hat{c}(k-1)\lambda_{X,2}^2\lambda_{S,2}^{-2} = 0, \end{aligned} \quad (70)$$

$$-\frac{1}{2\delta} + \hat{b}_1 + \hat{b}_2 = 0, \quad (71)$$

$$\hat{a}_1(d_1^{(k)} - \lambda_{S,1}^{(k)}) = 0, \quad (72)$$

$$\hat{a}_2(d_2^{(k)} - \lambda_{S,2}) = 0, \quad (73)$$

$$\hat{b}_1(\delta - ((d_1^{(k)})^{-1} + (\lambda_{S,1}^{(j)})^{-1} - (\lambda_{S,1}^{(k)})^{-1})^{-1}) = 0, \quad (74)$$

$$\begin{aligned} \hat{b}_2(\delta - ((d_2^{(k)})^{-1} + (\lambda_{S,1}^{(j)})^{-1} - \lambda_{S,2}^{-1})^{-1}) &= 0, \\ \hat{c}((\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2}d_1^{(k)} + \lambda_{X,1}^{(k)} - (\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-1} \\ + (k-1)(\lambda_{X,2}^2\lambda_{S,2}^{-2}d_2^{(k)} + \lambda_{X,2} - \lambda_{X,2}^2\lambda_{S,2}^{-1}) - kd_k) &= 0. \end{aligned} \quad (75)$$

Assume $d_k \in (d_{\min}^{(k)}, \gamma_X)$. It can be verified via algebraic manipulations that $\hat{\eta}(d_1^{(k)}, d_2^{(k)}, \delta) = \bar{r}(d_k)$ for

$$\begin{aligned} d_1^{(k)} &\triangleq ((\lambda_{S,1}^{(k)})^{-1} + (\lambda_Q^{(k)})^{-1})^{-1}, \\ d_2^{(k)} &\triangleq (\lambda_{S,2}^{-1} + (\lambda_Q^{(k)})^{-1})^{-1}, \\ \delta &\triangleq ((\lambda_{S,1}^{(j)})^{-1} + (\lambda_Q^{(k)})^{-1})^{-1}, \end{aligned} \quad (77)$$

where $\lambda_Q^{(k)}$ is given by (4). We shall identify the conditions under which this specific $(d_1^{(k)}, d_2^{(k)}, \delta)$ is a minimizer of $(\hat{\mathbf{P}})$. Clearly, (74)–(76) are satisfied. Moreover, in view of (72), (73) as well as the fact that $d_1^{(k)} < \lambda_{S,1}^{(k)}$ and $d_2^{(k)} < \lambda_{S,2}$, we must have

$$\hat{a}_m = 0, \quad m = 1, 2,$$

which, together with (69)–(71), implies

$$\begin{aligned} \hat{b}_1 &= \frac{\delta - d_1^{(k)} + 2k\hat{c}(\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2}(d_1^{(k)})^2}{2k\delta^2}, \\ \hat{b}_2 &= \frac{(k-1)(\delta - d_2^{(k)}) + 2k(k-1)\hat{c}\lambda_{X,2}^2\lambda_{S,2}^{-2}(d_2^{(k)})^2}{2k\delta^2}, \\ \hat{c} &= \frac{d_1^{(k)} + (k-1)d_2^{(k)}}{(\lambda_{X,1}^{(k)})^2(\lambda_{S,1}^{(k)})^{-2}(d_1^{(k)})^2 + (k-1)\lambda_{X,2}^2\lambda_{S,2}^{-2}(d_2^{(k)})^2} \\ &\quad \times \frac{1}{2k}. \end{aligned}$$

It is obvious that \hat{c} is nonnegative. Therefore, it suffices to have $\hat{b}_1 \geq 0$ and $\hat{b}_2 \geq 0$, which are equivalent to conditions (13) and (14), respectively (note that, when $j = k$, condition (13) is redundant and condition (14) is simplified to condition (12)). Moreover, under these conditions, every minimizer $(d_1^{(k)}, d_2^{(k)}, \delta)$ of $(\hat{\mathbf{P}})$ must satisfy (77) due to the fact that $\frac{1}{2} \log \frac{\lambda_{S,1}^{(j)}}{\delta}$ is a strictly convex function of δ (in other words, (63), (39), (40), (41), (64), (65), and (44) imply that δ is uniquely determined and is given by (77) when $r = \bar{r}(d_k)$). Hence, under conditions (13) and (14), when $r = \bar{r}(d_k)$, we can deduce $d_j \geq d_j^{(k)}(d_k)$ by substituting (77) into (68).

B. Degenerate Case

Subcase 1: $\lambda_{S,1}^{(j)} > \lambda_{S,2} = 0$. We have

$$\begin{aligned} \bar{r}^{(k)}(d_k) &= \frac{1}{2k} \log \frac{\gamma_X^2}{\gamma_S d_k - \gamma_X \gamma_Z}, \\ d_j^{(k)}(d_k) &= \frac{(j-k)\gamma_X^2 \gamma_Z + (k\gamma_S - j\gamma_Z)\gamma_X d_k}{(j\gamma_S - k\gamma_Z)\gamma_X - (j-k)\gamma_S d_k}. \end{aligned}$$

The desired conclusion that $r \geq \bar{r}^{(k)}(d_k)$ and that $d_j \geq d_j^{(k)}(d_k)$ when $r = \bar{r}^{(k)}(d_k)$ follows from the corresponding result for the quadratic Gaussian multiple description problem [26], [35]. Note that $(k-1)\lambda_{X,2}^2(\lambda_{S,1}^{(k)})\mu^{(k)}(\mu^{(k)} - 1) + k(\lambda_{X,1}^{(k)})^2\lambda_{S,2}^2 = 0$ (consequently, condition (11) is satisfied) when $\lambda_{S,1}^{(j)} > \lambda_{S,2} = 0$.

Subcase 2: $\lambda_{S,2} > \lambda_{S,1}^{(\ell)} = 0$. It can be verified that

$$\bar{r}^{(\ell)}(d_\ell) = \frac{\ell-1}{2\ell} \log \frac{(\ell-1)\lambda_{X,2}^2}{\ell\lambda_{S,2}d_\ell - (\ell-1)\lambda_{X,2}\lambda_{Z,2}},$$

which coincides with the rate-distortion function (normalized by ℓ) of the corresponding centralized remote source coding problem. Therefore, we must have $r \geq \bar{r}^{(\ell)}(d_\ell)$. Also, note that $(\lambda_{X,1}^{(\ell)})^2\lambda_{S,2}^2\nu^{(\ell)}(\nu^{(\ell)} - 1) + \ell\lambda_{X,2}^2(\lambda_{S,1}^{(\ell)})^2 = 0$ (consequently, condition (12) is satisfied for $k = \ell$) when $\lambda_{S,2} > \lambda_{S,1}^{(\ell)} = 0$.

V. CONCLUSION

We have studied the problem of robust distributed compression of correlated Gaussian sources in a symmetric setting and obtained a characterization of certain extremal points of the rate-distortion region. It is expected that one can make further progress by integrating our techniques with those developed for the quadratic Gaussian multiple description problem.

APPENDIX A

CALCULATION OF $d_{\min}^{(j)}$

Assuming $\Gamma_S^{(j)} \succ 0$ (i.e., $\lambda_{S,1}^{(j)} > 0$ and $\lambda_{S,2} > 0$), we have

$$\begin{aligned} &\sum_{i=1}^j \mathbb{E}[(X_i - \mathbb{E}[X_i|S_1, \dots, S_j])^2] \\ &= \text{tr}(\Gamma_X^{(j)} - \Gamma_X^{(j)}(\Gamma_S^{(j)})^{-1}\Gamma_X^{(j)}) \\ &= \text{tr}(\Lambda_X^{(j)} - \Lambda_X^{(j)}(\Lambda_S^{(j)})^{-1}\Lambda_X^{(j)}) \\ &= \frac{\lambda_{X,1}^{(j)}\lambda_{Z,1}^{(j)}}{\lambda_{S,1}^{(j)}} + (j-1)\frac{\lambda_{X,2}\lambda_{Z,2}}{\lambda_{S,2}}, \end{aligned}$$

from which the desired result follows immediately. The degenerate case $\lambda_{S,1}^{(j)} = 0$ or $\lambda_{S,2} = 0$ can be handled by performing the above analysis in a suitable subspace.

APPENDIX B

PROOF OF (23) AND (24)

For $t = 1, \dots, n$,

$$\begin{aligned} (G_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T &\triangleq (U_i(t))_{i \in \mathcal{A}}^T - \mathbb{E}[(U_i(t))_{i \in \mathcal{A}}^T | (S_i(t))_{i \in \mathcal{A}}^T] \\ &= (U_i(t))_{i \in \mathcal{A}}^T - \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}(S_i(t))_{i \in \mathcal{A}}^T, \end{aligned}$$

which is a k -dimensional zero-mean Gaussian random vector with covariance $\Gamma_U^{(k)} - \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}\Gamma_U^{(k)}$ and is independent of $(S_i^T)_{i \in \mathcal{A}}$. As a consequence,

$$(\hat{U}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T = \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}(\hat{S}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T, \quad t = 1, \dots, n.$$

Now it can be readily verified that

$$\begin{aligned} \Sigma_{\mathcal{A}}(t) &= \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}D_{\mathcal{A}}(t)(\Gamma_S^{(k)})^{-1}\Gamma_U^{(k)} \\ &\quad + \mathbb{E}[(G_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T (G_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}] \\ &= \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}D_{\mathcal{A}}(t)(\Gamma_S^{(k)})^{-1}\Gamma_U^{(k)} + \Gamma_U^{(k)} \\ &\quad - \Gamma_U^{(k)}(\Gamma_S^{(k)})^{-1}\Gamma_U^{(k)}, \quad t = 1, \dots, n, \end{aligned}$$

from which (23) follows immediately.

For $t = 1, \dots, n$, we have

$$\begin{aligned} \Delta_{\mathcal{A}}(t) &\leq \mathbb{E}[(S_i(t) - \tilde{S}'_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T (S_i(t) - \tilde{S}'_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}] \\ &= ((D_{\mathcal{A}}(t))^{-1} + (\Lambda_W^{(k)})^{-1} - (\Gamma_S^{(k)})^{-1})^{-1}, \quad (78) \end{aligned}$$

where $(\tilde{S}'_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T$ denotes the linear MMSE estimator of $(S_i(t))_{i \in \mathcal{A}}^T$ based on $(\hat{S}_{i,\mathcal{A}}(t))_{i \in \mathcal{A}}^T$ and $(U_i(t))_{i \in \mathcal{A}}^T$. Since $(A^{-1} + B^{-1})^{-1}$ is matrix concave in A for $A \succ 0$ and $B \succ 0$, it follows that

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n ((D_{\mathcal{A}}(t))^{-1} + (\Lambda_W^{(k)})^{-1} - (\Gamma_S^{(k)})^{-1})^{-1} \\ \leq (D_{\mathcal{A}}^{-1} + (\Lambda_W^{(k)})^{-1} - (\Gamma_S^{(k)})^{-1})^{-1}. \quad (79) \end{aligned}$$

Combining (78) and (79) proves (24).

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