Communication through a Finite-State Machine with Markov Property

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***Abstract*— We address the problem of communication through a finite-state machine with Markov property. The techniques from the theory of Markov decision processes are used to determine of the feedback capacity of this type of machines. We also consider the scenario that several users share a machine via TDMA. The capacity region for this scenario is established. Moreover, we adopt a game-theoretic viewpoint to interpret the operational meaning of the rate vectors in the capacity region.**

***Index Terms*— Game theory, Nash equilibrium, Markov decision processes, finite-state machine, feedback, TDMA.**

I. Introduction

Model 1 (Fig. 1) was first studied in [1]. It was shown in [1] that its feedback capacity and the optimal input policy can be computed via dynamic programming. Model 2 (Fig. 2) was first introduced in [2], in which, among other things, the feedback capacity was shown to be attainable by Markov input policy. The feedback capacity of Model 2 was determined in [3] for the binary output case and in [4] for the general case. It was realized in [4] that Model 1 can be converted to Model 2. So the results in [2-4] are applicable to Model 1. Here we show that Model 2 can also be converted to Model 1, i.e., Model 1 and Model 2 are equivalent. To facilitate the demonstration, we introduce a third model: Model 3 (Fig. 3) and show that all these three models are equivalent.

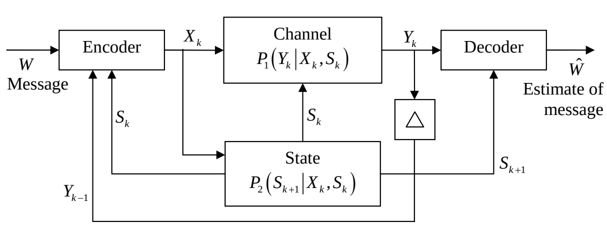

Fig. 1. Model 1.

Proposition 1: Model 1, 2 and 3 are equivalent.

Proof: Model 3 contains Model 1 since we can let $P\left(Y_k, S_{k+1} \middle| X_k, Y_{k-1}, S_k\right) = P_1\left(Y_k \middle| X_k, S_k\right) \cdot P_2\left(S_{k+1} \middle| X_k, S_k\right)$. Model 2 contains Model 3 since we can let $V_k = (Y_k, S_{k+1})$ and $Q\left(V_k \middle| X_k, V_{k-1}\right) = P\left(Y_k, S_{k+1} \middle| X_k, Y_{k-1}, S_k\right)$. Model 1 contains Model 2 since we can let $S_{k+1} = V_k$, $Q(V_k | X_k, V_{k-1}) = P\left(S_{k+1} \middle| X_k, S_k\right)$, and $\forall x_k, s_k$, $P_1(Y_k = c | X_k = x_k, S_k = s_k) = 1$ where c is a constant. ■

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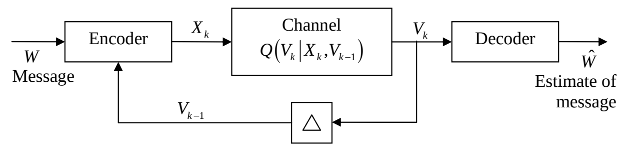

Fig. 2. Model 2.

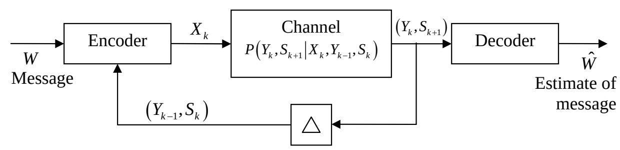

Fig. 3. Model 3.

Let x_k, y_k, s_k be the realization of X_k, Y_k and S_k respectively. Throughout this paper, we assume $x_k \in \mathcal{X}$, $y_k \in \mathcal{Y}$ and $s_k \in \mathcal{S}$ where $|\mathcal{X}|, |\mathcal{Y}|$ and $|\mathcal{S}|$ are finite. Without loss of generality, we supposes $\mathcal{X} = \{1, 2, \ldots, |\mathcal{X}|\}$, $\mathcal{Y} = \{1, 2, \ldots, |\mathcal{Y}|\}$ and $\mathcal{S} = \{1, 2, \ldots, |\mathcal{S}|\}$. Also, we only consider stationary machines, i.e., $P_1(\cdot|\cdot,\cdot)$ and $P_2(\cdot|\cdot,\cdot)$ (Model 1), $Q(\cdot|\cdot,\cdot)$ (Model 2) and $P(\cdot,\cdot|\cdot,\cdot,\cdot)$ (Model 3) do not depend on k.

In this paper, we mainly focus on Model 1 since it reveals more inner structure of finite-state machines.

The paper is organized as follows. In the next section, we give a detailed discussion of single user system. We determine its feedback capacity on the basis of the classification of Markov decision processes. We show that under a very general condition, the optimal information transmission scheme can be decomposed in two steps: 1. Control, 2. Communication; and we only need to design coding scheme for one initial state instead of developing different coding schemes for different initial states. In section III, we consider the scenario that several users share a finite-state machine via TDMA. We show that there exists a tradeoff among the information transmission rates of different users. We establish the capacity region for this multi-user TDMA system. A game-theoretic viewpoint is adopted to interpret the operational meaning of the rate vectors in the capacity region. Finally we discuss the biological implication of our model, which serves as the conclusion.

II. The Feedback Capacity of Single-User System

Definition 1: An (n, M, ϵ, s_1) feedback code function for Model 1 consists of

1) An encoding function f_{s_1} that maps the set of messages $\mathcal{W} = \{1, 2, \ldots, M\}$ to machine input words of block-length n through a sequence of functions $\{f_{s_1,k}\}_{k=1}^n$ that depend only on the message $W \in \mathcal{W}$, the machine outputs up to time $k-1$ and the machine states up to time k, i.e.,
$$X_k = f_{s_1,k}(W, Y_1^{k-1}, S_2^k) \tag{1}$$
2) A decoding function g_{s_1} that maps a received sequence of n pairs of machine output and machine state to the message set $g_{s_1} : \prod_1^n (\mathcal{Y} \times \mathcal{S}) \rightarrow \mathcal{W}$ such that the average probability of decoding error satisfies
$$P_e \triangleq \frac{1}{M} \sum_{w=1}^{M} P(\hat{W} \neq w | W = w, S_1 = s_1) \leqslant \varepsilon, \tag{2}$$
where $\hat{W} = g_{s_1}(Y_1^n, S_2^{n+1})$.

Note: Both the encoding function f_{s_1} and decoding function g_{s_1} depend on the initial states_1. Although it may seem to be more general to let the encoding function and decoding functions also depend on $y_{-\infty}^0, s_{-\infty}^0$, we will see later that this does not increases the capacity since s_1 is a sufficient statistic.

Definition 2: R_{s_1} is an ε-achievable rate given the initial state s_1 if for every $\delta > 0$ there exists, for all sufficiently large n, an (n, M, ε, s_1) code function such that $\frac{1}{n}\log M \geqslant R_{s_1} - \delta$. R_{s_1} is achievable if it is ε-achievable for all $\varepsilon > 0$. The supremum of all achievable rates R_{s_1} is defined as the feedback capacity $C_{s_1}^{fd}$ given the initial state s_1.

Theorem 1 (Converse Coding Theorem): Given the initial state s_1, information transmission with an arbitrary small expected frequency of errors is not possible if $R > \limsup_{n\to\infty} \frac{C_{s_1,n}}{n}$.

Here, $C_{s_1,n} = \max_{p(X_1^n)\in\mathcal{P}^*(X_1^n)} [I(X_1; Y_1, S_2 | S_1 = s_1) + \sum_{k=2}^{n} I(X_k; Y_k, S_{k+1}|S_k)]$ and $\mathcal{P}^*(X_1^n)$ is the set of Markov input policy, i.e., the conditional distribution on X_1^n of the form $P(X_k|X_1^{k-1}, Y_1^{k-1}, S_1^k) = P(X_k|S_k)$, $k = 1, 2, \ldots, n$.

Proof: Throughout the proof we implicitly assume that $P(S_1 = s_1) = 1$ and thus use S_1 instead of s_1.

Let W be the message random variable. By Fano's inequality,
$$H(W|Y_0^n, S_1^{n+1}) \leqslant h(P_e) + P_e \log M.$$
Since
$$\begin{aligned} H(W|Y_1^n, S_1^{n+1}) &= H(W) - I(W; Y_0^n, S_1^{n+1}) \\ &= \log M - I(W; Y_0^n, S_1^{n+1}), \end{aligned}$$
we have
$$(1 - P_e)\log M \leqslant h(P_e) + I(W; Y_0^n, S_1^{n+1}),$$
which we rewrite as
$$\frac{1}{n}\log M \leqslant \frac{h(P_e) + I(W; Y_0^n, S_1^{n+1})}{n(1 - P_e)}.$$
As $n \to \infty$, $P_e \to 0$. Hence, the feedback capacity
$$\begin{aligned} C_{s_1}^{fd} &= \limsup_{n\to\infty} \frac{1}{n}\log M \\ &\leqslant \limsup_{n\to\infty} \max_{p(X_1^n)} \frac{1}{n} I(W; Y_0^n, S_1^{n+1}). \end{aligned}$$
We have
$$\begin{aligned} I(W; Y_0^n, S_1^{n+1}) &= H(Y_0^n, S_1^{n+1}) \\ &\quad - H(Y_0^n, S_1^{n+1}|W) \\ &= \sum_{k=1}^{n} [H(Y_k, S_{k+1}|Y_0^{k-1}, S_1^k) \\ &\quad - H(Y_k, S_{k+1}|W, Y_0^{k-1}, S_1^k)] \\ &\leqslant \sum_{k=1}^{n} [H(Y_k, S_{k+1}|S_k) \\ &\quad - H(Y_k, S_{k+1}|W, X_k, Y_0^{k-1}, S_1^k)] \\ &\overset{(a)}{=} \sum_{k=1}^{n} [H(Y_k, S_{k+1}|S_k) \\ &\quad - H(Y_k, S_{k+1}|X_k, S_k)] \\ &= \sum_{k=1}^{n} I(X_k; Y_k, S_{k+1}|S_k) \end{aligned} \tag{3}$$
where (a) holds because, when conditioned on the input X_k and the current state S_k, the output Y_k and next state S_{k+1} become independent of the message W, the earlier outputs Y_0^{k-1} and the earlier states S_1^{k-1}. Hence we have
$$\begin{aligned} C_{s_1}^{fd} &\leqslant \limsup_{n\to\infty} \max_{p(X_1^n)} \frac{1}{n} I(W; Y_0^n, S_1^{n+1}) \\ &\leqslant \limsup_{n\to\infty} \max_{p(X_1^n)} \frac{1}{n} \sum_{k=1}^{n} I(X_k; Y_k, S_{k+1}|S_k) \\ &\overset{(b)}{=} \limsup_{n\to\infty} \max_{p(X_1^n)\in\mathcal{P}^*(X_1^n)} \frac{1}{n} \sum_{k=1}^{n} I(X_k; Y_k, S_{k+1}|S_k) \end{aligned} \tag{4}$$
where (b) follows by the dominance of Markov policy, see e.g. [5]. ■

Note: From the above theorem, we can see that there is no loss of generality to search for the optimal input policy inside the set of Markov policies, i.e., X_k only needs to depend only on S_k. The feedback Y_k is thus useless.

Now we begin to compute $\limsup_{n\to\infty} \frac{C_{s_1,n}}{n}$ on the basis of the classification of Markov decision processes. See [5] for the detailed discussion of the classification schemes.

(*i*) *Weak communicating* (there exists a closed set of states $\mathcal{S}' \subseteq \mathcal{S}$, with each state in $\mathcal{S}'$ accessible from every other state in that set under some deterministic stationary input policy, plus a possibly empty set of states which is transient under every input policy):

$$\limsup_{n\to\infty} \frac{C_{s_1,n}}{n} = \max_{p(X_k|S_k)\in\mathcal{P}^{**}} \sum_{s\in\mathcal{S}'} \mu_s I(X_k;Y_k,S_{k+1}|S_k=s). \quad (5)$$

where $\mathcal{P}^{**}$ is the set of all stationary Markov input policies under which there is a single ergodic chain in the state space $\mathcal{S}$ and $\{\mu_s\}_{s\in\mathcal{S}'}$ is the induced stationary distribution of $\{S_k, k=1,2,\ldots\}$ on $\mathcal{S}'$.
Note: this result has been obtained partially in [3, 4] and implicitly in [1].

(*ii*) *Multichain* (the transition matrix corresponding to at least one stationary policy contains two or more closed irreducible recurrent classes): It is possible but rather intricate to determine $\limsup\limits_{n\to\infty} \frac{C_{s_1,n}}{n}$ in this class. The derivation is thus omitted. We just mention that in general multichain model can be decomposed into several disjoint communicating model.

Since (5) come from the converse coding theorem, it is an upper bound on $C^{fd}_{s_1}$. But actually (5) is achievable if our model is in Class (*i*). This follows from evaluating the general feedback capacity formula in [1]. So $C^{fd}_{s_1}$ is determined if our model is in Class (*i*). It is interesting to see that in Class (*i*) , $C^{fd}_{s_1}$ does not depend on s_1. Actually this phenomenon can be explained by an intuitive argument which is stated in the following theorem.

Theorem 2: Let P^π_i denote the probability measure induced by the initial state $S_1 = i$, the input policy π and the conditional probability associated with the machine. Let $\alpha_j = \min\{k : S_k = j, k = 1,2,\ldots\}$. If there exists a input policy π under which $P^\pi_i(\alpha_j < \infty) = 1$, then $C^{fd}_i \geqslant C^{fd}_j$. (Note: this theorem is not restricted to our machine model.)

Proof: (Sketch) We can first drive the machine state from i to j, which can be done within finite steps with probablity one, and then use the optimal coding scheme designed for initial state j. As $n\to\infty$, such a strategy asymptotically achieves rate C^{fd}_j. Since this strategy is not necessarily optimal for initial state i, we can conclude that $C^{fd}_i \geq C^{fd}_j$. ■

The above theorem suggests that in general the transmission scheme can be decomposed into two steps. Assume $C^{fd}_{\hat{s}} = \max\limits_{s\in\mathcal{S}} C^{fd}_s$ and from any $s\in\mathcal{S}$, there exists an input policy under which the machine can be driven to state $\hat{s}$ using finite steps with probability one (Note: by Theorem 2 this assumption actually implies that $C^{fd}_{s'} = C^{fd}_{s''} \quad \forall s', s'' \in \mathcal{S}$). So if the initial state is not $\hat{s}$, we can first drive the machine to state $\hat{s}$ and then use the optimal coding scheme designed for the initial state $\hat{s}$. The advantage of doing this is that now we only need to design a coding scheme for one initial state instead of developing different coding schemes for different initial states.

Now let's return to (5) to get an intuitive feeling. It's well-known [6, 7] that when the state process is ergodic and independent of the input and output processes,

$$C^{fd}_{s_1} = \sum_{s=1}^{|\mathcal{S}|} \mu_s \max_{p(X|S=s)} I(X;Y|S=s), \quad \forall s_1 \in \mathcal{S} \qquad (6)$$

It's easy to see that (6) can be reduced from (5) if we let $P_2(S_{k+1}|X_k,S_k) = P_2(S_{k+1}|S_k)$ and let Markov process $\{S_k, k=1,2,\ldots\}$ be irreducible. The difference between (5) and (6) suggests that

(*a*) when the state process is unaffected by the input and output processes, we should maximize the mutual information for each state;

(*b*) when the state process can be affected by the input and/or output process, the greedy method in (*a*) is generally not optimal since by maximizing the mutual information for each state, the machine may be driven to the states with low mutual information too frequently. So the joint optimization (i.e., optimize the mutual information for each state and the induced stationary distribution jointly) should be used.

Let's consider the following example (Fig. 4).

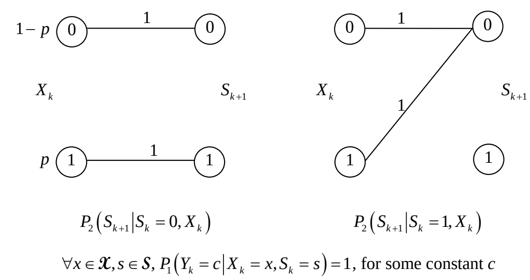

Fig. 4. Example.

Let $P(X_k = 1|S_k = 0) = p$, then the equilibrium distribution of $\{S_k, k=1,2,\ldots\}$ is $(\mu_0,\mu_1) = \left(\frac{1}{1+p}, \frac{p}{1+p}\right)$ and the induced transmission rate is $\frac{1}{1+p}h(p)$ which is maximized when $p = p^* = \frac{3-\sqrt{5}}{2} \approx 0.38$ with corresponding value $\frac{1}{1+p^*}h(p^*) \approx 0.69$ bits per machine use. Note:$p^* < \frac{1}{2}$. The information rate induced by setting $p = \frac{1}{2}$ is $\frac{2}{3}$ (bits per machine use) which is less than 0.69 (bits per machine use).

III. TDMA Multi-User System

In this section, we consider the model shown in Fig. 5.

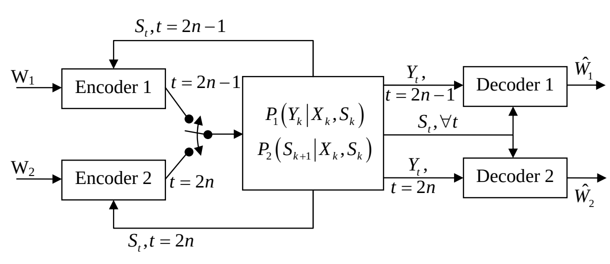

Fig. 5. Two-user system.

Two transmitter-receiver pairs share a finite-state machine via TDMA. Transmitter i wants to convey message W_i to Receiver i, $i = 1, 2$. W_1 and W_2 are assumed to be independent. Transmitter 1 can use the machine in the odd time slots while Transmitter 2 can use the machine in the even time slots. Each transmitter observes, in a causal way, the realization of the machine state process in its transmission slots. Receiver 1 tries to recover message W_1 based on all the machine outputs in the odd time slots and all the machine states upon the decoding time. Receiver 2 tries to recover message W_2 based on all the machine outputs in the even time slots and all the machine states upon the decoding time. Except through observing the machine state realization in their transmission slot, Transmitter 1 and Transmitter 2 are not allowed to convene. So one does not know beforehand what is the message that the other wants to transmit. Now the question is what are the rate pairs that these two users can achieve. Here we regard a transmitter-receiver pair as a user.

This question is completely trivial when the machine state process is ergodic and unaffected by the machine input and output processes. In that case, since there is no interference, each user can achieve the half capacity of the single-user system but no more. So the capacity region is given by (see Fig. 6)

$$R_i \leqslant \frac{1}{2}C \quad i = 1, 2, \tag{7}$$

where $C = \sum_{s=1}^{|\mathcal{S}|} \mu_s \max_{p(X|S=s)} I(X; Y|S = s)$ is the feedback capacity of single-user system.

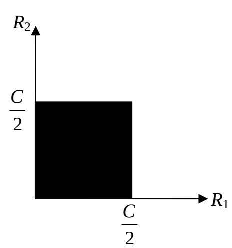

Fig. 6. Capacity region for two-user TDMA system with no interference.

But when the state process can be affected by the input, the above result does not hold anymore. Let's consider the two-user system with the machine specified by Fig. 4. Since when $S_k = 1$, S_{k+1} is always zero no matter what the input is, we only need to specify the input policies of User 1 and 2 when $S_k = 0$. Suppose User 1 chooses the policy π_1:$P(X_k = 0|S_k = 0) = P(X_k = 1|S_k = 0) = \frac{1}{2} \quad (k = 1, 3, \ldots)$ while User 2 chooses the policy π_2:$P(X_k = 0|S_k = 0) = 1$ $(k = 2, 4, \ldots)$. Clearly, under π_1 and π_2, $P(S_k = 0, k = 3, 5, \ldots) = 1$. So Transmitter 1 always faces the good state (i.e., state 0) and can transmit 1 bit of information in each of its transmission slot while Transmitter 2 can not transmit any information at all. So under π_1 and π_2, the rate pair $\left(\frac{1}{2}, 0\right)$ is achievable. Since the role of User 1 and User 2 can be interchanged, the rate pair $\left(0, \frac{1}{2}\right)$ is also achievable. It is interesting to note that $\frac{1}{2} > \frac{0.694}{2}$, i.e., although only transmitting half of time, one of the users can achieve the transmission rate higher than half of the capacity of the single-user case, which is fundamentally different from (7). Of course, this is obtained by sacrificing the transmission rate of the other user. Now a natural question is to ask whether there exist some other rate pairs that are also achievable.

Let's still consider the previous example. Suppose User 1 chooses the policy $\pi_1(p)$: $P(X_k = 1|S_k = 0) = p \quad (k = 1, 3, \ldots)$ while User 2 chooses the policy $\pi_2(q)$:$P(X_k = 1|S_k = 0) = q \quad (k = 2, 4, \ldots)$. Under $\pi_1(p)$ and $\pi_2(q)$, the state process $\{S_k\}_{k=1}^{\infty}$ is in general a nonhomogeneous Markov chain since the state transition matrix $P\left(S_{k+1} \mid S_k\right)$ is $T_1 = \begin{bmatrix} 1-p & 1 \\ p & 0 \end{bmatrix}$ when k is odd and is $T_2 = \begin{bmatrix} 1-q & 1 \\ q & 0 \end{bmatrix}$ when k is even. But it's interesting to see that $\{S_{2k-1}\}_{k=1}^{\infty}$ is a homogeneous Markov chain with transition matrix $T_2T_1 = \begin{bmatrix} 1-q+pq & 1-q \\ q(1-p) & q \end{bmatrix}$ and $\{S_{2k}\}_{k=1}^{\infty}$ is a homogeneous Markov chain with transition matrix $T_1T_2 = \begin{bmatrix} 1-p+pq & 1-p \\ p(1-q) & p \end{bmatrix}$. Because matrix multiplication is not commutative, $T_1T_2 \neq T_2T_1$ in general. $\{S_{2k-1}\}_{k=1}^{\infty}$ is irreducible when $p, q \neq 1$ and has a unique stationary distribution $(\mu_0', \mu_1') = \left(\frac{1-q}{1-qp}, \frac{q(1-p)}{1-qp}\right)$.$\{S_{2k}\}_{k=1}^{\infty}$ is irreducible when $p, q \neq 1$ and has a unique stationary distribution $(\mu_0'', \mu_1'') = \left(\frac{1-p}{1-qp}, \frac{p(1-q)}{1-qp}\right)$. So the limiting average transmission rate of User 1 is $\frac{1}{2}\mu_0' h\left(p\right)$ (per machine use) while the limiting average transmission rate of User 2 is $\frac{1}{2}\mu_0'' h\left(q\right)$ (per machine use). This gives the following achievable rate pair:

$$\left(\frac{1}{2}\frac{1-q}{1-qp}h\left(p\right), \frac{1}{2}\frac{1-p}{1-qp}h\left(q\right)\right)$$

which is plotted in Fig. 7

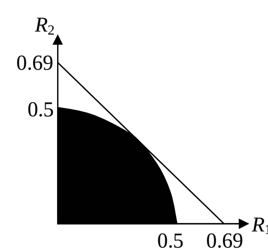

Fig. 7. An achievable region for two-user TDMA system with interference

This example suggests that for the general case, the following rate region:

$$\begin{aligned} &\{(R_1, R_2) : \\ &0 \leq R_i \leq \frac{1}{2}\sum_{s=1}^{|S|} \mu_{i,s}(\vec{p}_1, \vec{p}_2) I_{p_{i,s}}(X_k; Y_k, S_{k+1}|S_k = s) \\ &\qquad\qquad\qquad i = 1, 2, \quad \forall \vec{p}_1, \vec{p}_2\} \end{aligned} \tag{8}$$

is achievable. Here

1) $\vec{p}_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,|\mathcal{S}|})$ $(i = 1, 2)$ with each entry being a probability measure on $\mathcal{X}$;
2) $\vec{\mu}_i(\vec{p}_1, \vec{p}_2) = (\mu_{i,1}(\vec{p}_1, \vec{p}_2)$, $\mu_{i,2}(\vec{p}_1, \vec{p}_2)$, $\ldots$,$\mu_{i,|S|}(\vec{p}_1, \vec{p}_2))$ $(i = 1, 2)$ is the stationary distribution of the Markov chain governed by the transition matrix $T_{(i \bmod 2)+1}T_i$, where T_m $(m = 1, 2)$ is an $|\mathcal{S}| \times |\mathcal{S}|$ matrix whose (i, j) entry is
$$\sum_{l=1}^{|\mathcal{X}|} P_2(S_{k+1} = i | X_k = l, S_k = j) p_{m,j}(X_k = l) \quad (9)$$
3) $I_{p_{i,s}}(X_k; Y_k, S_{k+1} | S_k = s)$ is the mutual information between X_k and (Y_k, S_{k+1}) when the current state is s and the input distribution is $p_{i,s}$.

But there are some technical problems since under certain input policy, the Markov chain governed by the transition matrix T_1T_2 or T_2T_1 may not be irreducible and it's in general not easy to check. Here we give a sufficient condition under which the Markov chain governed by the finite product of transition matrix is always irreducible no matter what the input distribution is.

Definition 3 (Strong irreducibility): Let

$$\tilde{T}(i, j) = \min_{l \in \{1,2,\ldots|\mathcal{X}|\}} P_2(S_{k+1} = j | X_k = l, S_k = i)$$

We say there exists a directed edge from state i to state j if $\tilde{T}(i, j) > 0$. We say a Markov chain $\{S_k, k = 1, 2, \ldots\}$ is strongly irreducible if for any two states i and j (i can be equal to$j$), there exists a directed path from i to j. For simplicity, we just say $\tilde{T}$, the $|\mathcal{S}| \times |\mathcal{S}|$ matrix whose (i, j) entry is $\tilde{T}(i, j)$, is strongly irreducible, since $\tilde{T}$ contains all the information that determines whether the Markov chain $\{S_k, k = 1, 2, \ldots\}$ is strongly irreducible or not.

Definition 4 (Strong aperiodicity): Where the "length" of a path is the number of edges comprising the path, let $\mathcal{D}_i$ be the set of lengths of all the possible closed paths from state i to state i. Let d_i be the greatest common divisor of $\mathcal{D}_i$. d_i is called the period of state i.

The following result says that period is a class property.

Lemma 1: If the Markov chain $\{S_k, k = 1, 2, \ldots\}$ is strongly irreducible, then $d_i = d_j$ for any i and j. (See [4].)

So for a strongly irreducible Markov chain $\{S_k, k = 1, 2, \ldots\}$, all the states have the same period, which we shall by d. We say a strongly irreducible Markov chain $\{S_k, k = 1, 2, \ldots\}$is strongly aperiodic if $d = 1$. For simplicity, we just say that $\tilde{T}$ is strongly irreducible and strongly aperiodic.

Definition 5: For any two matrices $A = (a_{i,j}) \in \mathcal{R}^{m\times n}$,$B = (b_{i,j}) \in \mathcal{R}^{m\times n}$, we say $A \geqslant B$ if $a_{i,j} \geqslant b_{i,j}$ for all i and j.

Theorem 3: For any positive integer m, and input distribution $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m$, define T_i, $i = 1, 2, \ldots, m$ as in (9). If $\tilde{T}$ is strongly irreducible with period d and $\mathrm{GCD}\,(m, d) = 1$, then the Markov chain governed by the transition matrix $\prod_{i=1}^{m} T_i$ is irreducible.

Proof: It's easy to see that $\prod_{i=1}^{m} T_i \geqslant \tilde{T}^m$. Since $\tilde{T}$ is strongly irreducible, every row of $\tilde{T}$ should have at least one positive element. So we can scale every row of $\tilde{T}$ to make it to be a transition matrix T in which the summation of the elements on every row is 1. Clearly, T is irreducible with the same period as $\tilde{T}$ and we have $T^m(i, j) = 0 \Leftrightarrow \tilde{T}^m(i, j) = 0$.

Given a $r \times r$ real matrix M, we say a directed graph G with r vertices is generated by M if $M(i, j) > 0 \Leftrightarrow$ there is a directed edge from vertex i to vertex j. So the graph generated by T^m is identical to the graph generated by $\tilde{T}^m$. Since $\prod_{i=1}^{m} T_i \geqslant \tilde{T}^m$, the graph generated by $\tilde{T}^m$ is a subgraph of the one generated by $\prod_{i=1}^{m} T_i$. Since the Markov chain governed by the transition matrix T is irreducible with period d and $\mathrm{GCD}\,(m, d) = 1$, the Markov chain governed by the transition matrix T^m is also reducible. Since the irreducibility of a Markov chain is fully determined by the directed graph generated by its transition matrix, we can say a directed graph is irreducible if its associated Markov chain is. It's easy to see that a directed graph is irreducible if it contains an irreducible subgraph whose vertex set is same as the original graph. Hence, the graph generated by $\prod_{i=1}^{m} T_i$ is irreducible and we can conclude that the Markov chain governed by the transition matrix $\prod_{i=1}^{m} T_i$ is irreducible. ■

Corollary 1: For any positive integer m, any input distribution $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m$, define T_i, $i = 1, 2, \ldots, m$ as in (9). The Markov chain governed by the transition matrix $\prod_{i=1}^{m} T_i$ is irreducible and aperiodic if $\tilde{T}$ is strongly irreducible and strongly aperiodic.

Proof: Since $\tilde{T}$ is a periodic, by Definition 4, we have $d = 1$. The "irreducible" part in the corollary now follows by Theorem 3 since $\mathrm{GCD}\,(m, d) = \mathrm{GCD}\,(m, 1) = 1$. For the "aperiodic" part, observe that $\tilde{T}$ is strongly aperiodic $\Rightarrow T$ is aperiodic $\Rightarrow T^m$ is aperiodic $\Rightarrow \prod_{i=1}^{m} T_i$ is aperiodic. ■

By Theorem 3, (8) can be generalized to m-user case.

Although the analysis in this section is rather heuristic, it can be converted into a rigorous random coding argument. Furthermore, since the capacity region $\mathcal{C}$ is convex and closed, we have

$$\mathcal{C} = \bigcap_{0 \le \lambda \le 1} \{(R_1, R_2) : \lambda R_1 + (1 - \lambda) R_2 \le \Sigma_\lambda\}$$

where $\Sigma_\lambda = \max_{(R_1,R_2)\in\mathcal{C}} \lambda R_1 + (1 - \lambda) R_2$.
Σ_λ can be computed in a way similar to that in Theorem 1. It turns out that the achievable rate region given by (8) is exactly the capacity region. See [8] for the details.

From the above analysis, we can see that the procedure to find the capacity region, especially to find the boundary of the capacity region, is same as those standard procedures in the stochastic game theory. Actually our problem can be converted into the following form:

In a stochastic game with two players, where the reward function is the sum of weighted long term directed mutual information of each player, what kind of input policies should these two players choose in order to maximize the reward function?

In the above, we assume that two users will cooperate since they have a single objective. A natural question is to ask if the individual users have objectives that are in conflict with each other, e.g. each user only cares about the transmission rate of himself, whether they will still cooperate. The answer is "Yes". The rigorous analysis is omitted due to the page count constraint. Interested readers could see [8] for the details. A similar problem in the setting of the Gaussian multiaccess channel is addressed in [9].

We can also address the problem from the viewpoint of non-cooperative game theory. For simplicity, here we only discuss a simple example. The conclusion actually holds for much more general setting. Again let's consider the two-user system with the machine specified by Fig.4. We know that if User 1 chooses the policy π_1:$P(X_k = 1|S_k = 0) = p \quad (k = 1, 3, \dots)$ and User 2 chooses the policy π_2:$P(X_k = 1|S_k = 0) = q$ $(k = 2, 4, \dots)$, then the corresponding achievable rate pair is

$$\left(\frac{1}{2}\frac{1-q}{1-qp}h\left(p\right), \frac{1}{2}\frac{1-p}{1-qp}h\left(q\right) \right)$$

By Brower's fixed point theorem, there exists p^* and q^* such that

$$\frac{1}{2}\frac{1-q^*}{1-q^*p^*}h\left(p^*\right) = \max_p \left\{ \frac{1}{2}\frac{1-q^*}{1-q^*p}h\left(p\right) \right\}$$

$$\frac{1}{2}\frac{1-p^*}{1-q^*p^*}h\left(q^*\right) = \max_q \left\{ \frac{1}{2}\frac{1-p^*}{1-qp^*}h\left(q\right) \right\}$$

Here the policy pair (p^*, q^*) may not be unique. For each p^* and q^*, the associated rate pair is a Nash equilibrium for rate allocation. By numerical method, we can get that for our example, there are two solutions: $p^* = q^* = 1$ and $p^* = q^* = 0.696$. The resulting rate pairs are $(R_1^* = 0, R_2^* = 0)$ and $(R_1^* = 0.261, R_2^* = 0.261)$ respectively. See Fig. 8.

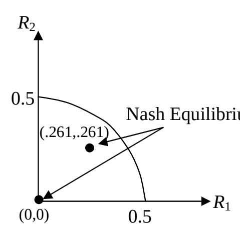

Fig. 8. Nash equilibrium in the capacity region.

The operational meaning of the Nash-equilibrium rate pair and the corresponding coding scheme is that:

If one user adheres to the current coding scheme, then it's impossible for the other user to achieve the reliable communication at a rate higher than that supported by the current coding scheme.

This is important in the interference communication scenario. Suppose two users agree to communicate at a rate pair on the boundary of the capacity region and one user adheres to the contract. But the other user may secretly break the contract and design a coding scheme which can support a transmission rate higher than his current one. This action may hurt the communication performance of the user who adheres to the contract. However, if the agreement is made at the Nash-equilibrium rate pair, then the user that adheres to the contract don't need to worry since the other user is not able to increase his communication rate by breaking the contract and thus will not have the incentive to do so. Note also that we assume both users are selfish but not evil, i.e., the objective of each user is to maximize his own transmission rate, not to deprive the transmission rate of the other.

IV. Conclusion

Our model requires that both transmitter and receiver know the state information, which seems unrealistic for most of real communication systems. But in some biological systems, this assumption can be justified. See Fig. 9

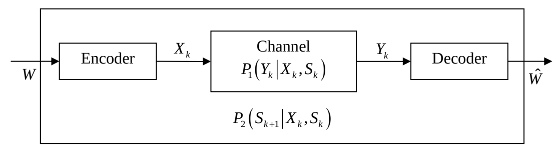

Fig. 9. A communication model for a biological system

The main feature of Fig. 9 is that both the encoder and decoder are inside the machine. In such a case, the state of the machine is not only the state of the channel between the encoder and the decoder, but also the state of the encoder and decoder themselves. We can imagine that when such a system is well-designed, which is fulfilled by evolution and natural selection for biological system, the encoder and the decoder can be matched to the channel between them.

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