From Integrated Electronics, by Millman & Halkias. McGraw Hill 1972

necessary to approximate it. An approximation for an ideal low-pass filter is of the form

$$A_V(s) = \frac{1}{P_n(s)} \tag{16-17}$$

where $P_n(s)$ is a polynomial in the variable s with zeros in the left-hand plane. Active filters permit the realization of arbitrary left-hand poles for $A_V(s)$, using the operational amplifier as the active element and only resistors and capacitors for the passive elements.

Since commercially available OP AMPS have unity gain-bandwidth products as high as 100 MHz, it is possible to design active filters up to frequencies of several MHz. The limiting factor for full-power response at those high frequencies is the slewing rate (Sec. 15-6) of the operational amplifier. (Commercial integrated OP AMPS are available with slewing rates as high as 100 V/μs.)

Butterworth Filter[6] A common approximation of Eq. (16-17) uses the Butterworth polynomials $B_n(s)$, where

$$A_V(s) = \frac{A_{Vo}}{B_n(s)} \tag{16-18}$$

and with $s = j\omega$,

$$|A_V(s)|^2 = |A_V(s)|\,|A_V(-s)| = \frac{A_{Vo}^2}{1 + (\omega/\omega_o)^{2n}} \tag{16-19}$$

From Eqs. (16-18) and (16-19) we note that the magnitude of $B_n(\omega)$ is given by

$$|B_n(\omega)| = \sqrt{1 + \left(\frac{\omega}{\omega_o}\right)^{2n}} \tag{16-20}$$

The Butterworth response [Eq. (16-19)] for various values of n is plotted in Fig. 16-16. Note that the magnitude of A_V is down 3 dB at $\omega = \omega_o$ for all n. The larger the value of n, the more closely the curve approximates the ideal low-pass response of Fig. 16-15a.

If we normalize the frequency by assuming $\omega_o = 1$ rad/s, then Table 16-1 gives the Butterworth polynomials for n up to 8. Note that for n even, the polynomials are the products of quadratic forms, and for n odd, there is present the additional factor $s + 1$. The zeros of the normalized Butterworth polynomials are either -1 or complex conjugate and are found on the so-called *Butterworth circle* of unit radius shown in Fig. 16-17. The *damping factor* k is defined as one-half the coefficient of s in each quadratic factor in Table 16-1. For example, for $n = 4$, there are two damping factors, namely, $0.765/2 = 0.383$ and $1.848/2 = 0.924$. It turns out (Prob. 16-20) that k is given by

$$k = \cos\theta \tag{16-21}$$

where θ is as defined in Fig. 16-17a for n even and Fig. 16-17b for n odd.

TABLE 16-1 Normalized Butterworth polyhominals

n	Factors of polynomial $P_n(s)$
1	$(s + 1)$
2	$(s^2 + 1.414s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$
5	$(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + 1.414s + 1)(s^2 + 1.932s + 1)$
7	$(s + 1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$
8	$(s^2 + 0.390s + 1)(s^2 + 1.111s + 1)(s^2 + 1.663s + 1)(s^2 + 1.962s + 1)$

From the table and Eq. (16-18) we see that the typical second-order Butterworth filter transfer function is of the form

$$\frac{A_V(s)}{A_{Vo}} = \frac{1}{(s/\omega_o)^2 + 2k(s/\omega_o) + 1} \tag{16-22}$$

where $\omega_o = 2\pi f_o$ is the high-frequency 3-dB point. Similarly, the first-order filter is

$$\frac{A_V(s)}{A_{Vo}} = \frac{1}{s/\omega_o + 1} \tag{16-23}$$

Practical Realization Consider the circuit shown in Fig. 16-18*a*, where the active element is an operational amplifier whose stable midband gain

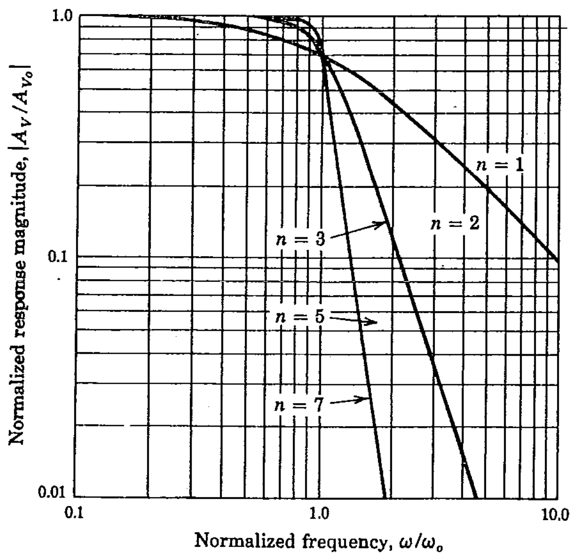

Fig. 16-16 Butterworth low-pass-filter frequency response.

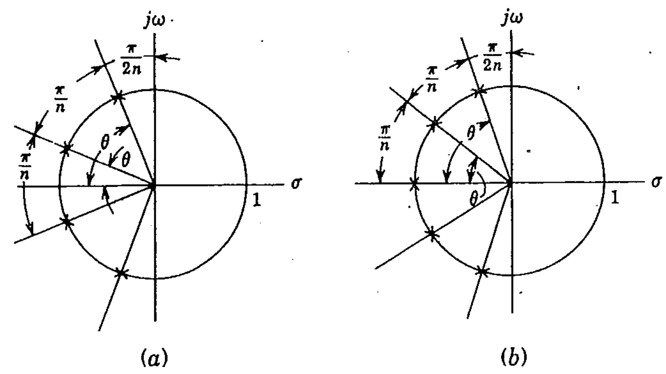

Fig. 16-17 The Butterworth circle for (*a*) n even and (*b*) n odd. Note that for n odd, one of the zeros is at $s = -1$.

$V_o/V_i = A_{Vo} = (R_1 + R_1')/R_1$ [Eq. (15-4)] is to be determined. We assume that the amplifier input current is zero, and we show in Prob. 16-25 that

$$A_V(s) = \frac{V_o}{V_s} = \frac{A_{Vo}Z_3Z_4}{Z_3(Z_1 + Z_2 + Z_3) + Z_1Z_2 + Z_1Z_4(1 - A_{Vo})} \tag{16-24}$$

If this network is to be a low-pass filter, then Z_1 and Z_2 are resistances and Z_3 and Z_4 are capacitances. Let us assume $Z_1 = Z_2 = R$ and $C_3 = C_4 = C$, as shown in Fig. 16-18*b*. The transfer function of this network takes the form

$$A_V(s) = A_{Vo}\frac{(1/RC)^2}{s^2 + \left(\dfrac{3 - A_{Vo}}{RC}\right)s + \left(\dfrac{1}{RC}\right)^2} \tag{16-25}$$

Comparing Eq. (16-25) with Eq. (16-22), we find

$$\omega_o = \frac{1}{RC} \tag{16-26}$$

and

$$2k = 3 - A_{Vo} \qquad \text{or} \qquad A_{Vo} = 3 - 2k \tag{16-27}$$

We are now in a position to synthesize even-order Butterworth filters by cascading prototypes of the form shown in Fig. 16-18*b*, using identical R's and C's and selecting the gain A_{Vo} of each operational amplifier to satisfy Eq. (16-27) and the damping factors from Table 16-1.

To realize odd-order filters, it is necessary to cascade the first-order filter of Eq. (16-23) with second-order sections such as indicated in Fig. 16-18*b*. The first-order prototype of Fig. 16-18*c* has the transfer function of Eq. (16-23) for arbitrary A_{Vo} provided that ω_o is given by Eq. (16-26). For example, a third-order Butterworth active filter consists of the circuit in Fig. 16-18*b* in cascade with the circuit of Fig. 16-18*c*, with R and C chosen so that $RC = 1/\omega_o$, with A_{Vo} in Fig. 16-18*b* selected to give $k = 0.5$ (Table 16-1, $n = 3$), and A_{Vo} in Fig. 16-18*c* chosen arbitrarily.

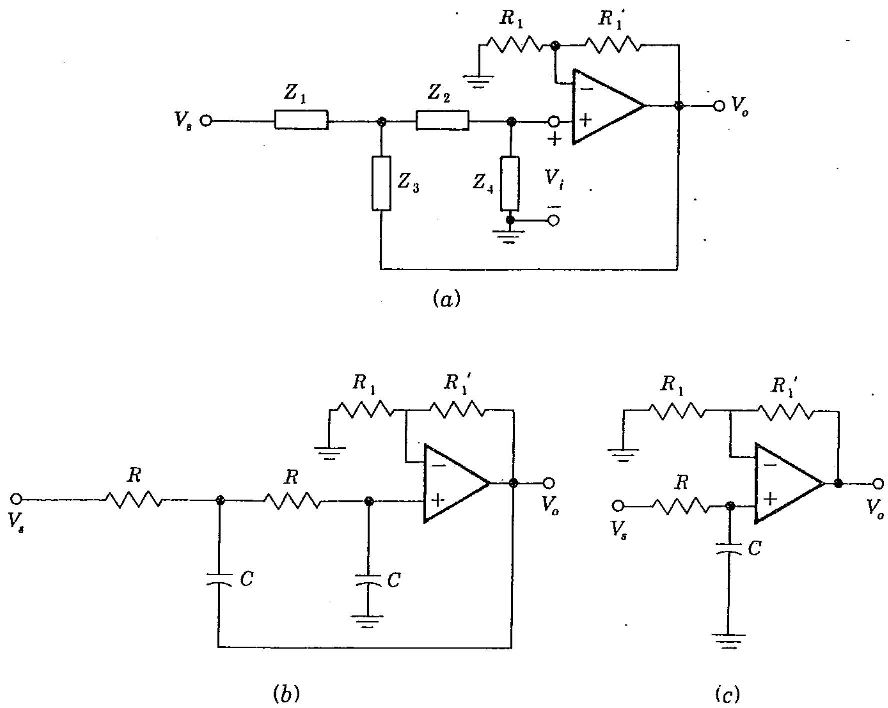

Fig. 16-18 (a) Generalized active-filter prototype. (b) Second-order low-pass section. (c) First-order low-pass section.

EXAMPLE Design a fourth-order Butterworth low-pass filter with a cutoff frequency of 1 kHz.

Solution We cascade two second-order prototypes as shown in Fig. 16-19. For $n = 4$ we have from Table 16-1 and Eq. (16-27)

$$A_{V1} = 3 - 2k_1 = 3 - 0.765 = 2.235$$

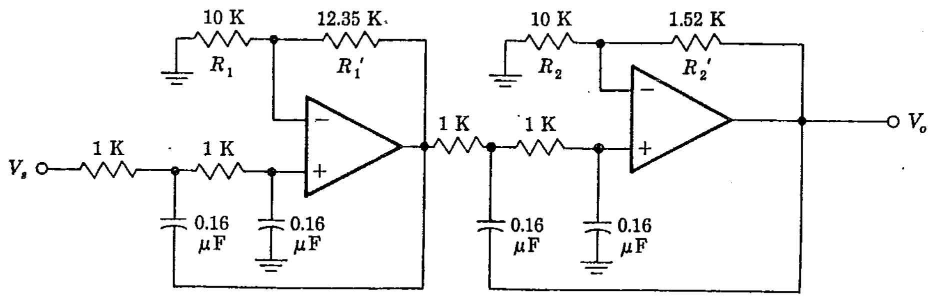

Fig. 16-19 Fourth-order Butterworth low-pass filter with f_o = 1 kHz.

and

$$A_{V2} = 3 - 2k_2 = 3 - 1.848 = 1.152$$

From Eq. (15-4), $A_{V1} = (R_1 + R_1')/R_1$. If we arbitrarily choose $R_1 = 10$ K, then for $A_{V1} = 2.235$, we find $R_1' = 12.35$ K, whereas for $A_{V2} = 1.152$, we find $R_2' = 1.520$ K and $R_2 = 10$ K. To satisfy the cutoff-frequency requirement, we have, from Eq. (16-26), $f_o = 1/2\pi RC$. We take $R = 1$ K and find $C = 0.16\ \mu$F. Figure 16-19 shows the complete fourth-order low-pass Butterworth filter.

High-pass Prototype An idealized high-pass-filter characteristic is indicated in Fig. 16-15*b*. The high-pass second-order filter is obtained from the low-pass second-order prototype of Eq. (16-22) by applying the transformation

$$\left.\frac{s}{\omega_o}\right|_{\text{low-pass}} \rightarrow \left.\frac{\omega_o}{s}\right|_{\text{high-pass}} \qquad (16\text{-}28)$$

Thus, interchanging R's and C's in Fig. 16-18*b* results in a second-order high-pass active filter.

Bandpass Filter A second-order bandpass prototype is obtained by cascading a low-pass second-order section whose cutoff frequency is f_{oH} with a high-pass second-order section whose cutoff frequency is f_{oL}, provided $f_{oH} > f_{oL}$, as indicated in Fig. 16-15*c*.

Band-reject Filter Figure 16-20 shows that a band-reject filter is obtained by paralleling a high-pass section whose cutoff frequency is f_{oL} with a low-pass

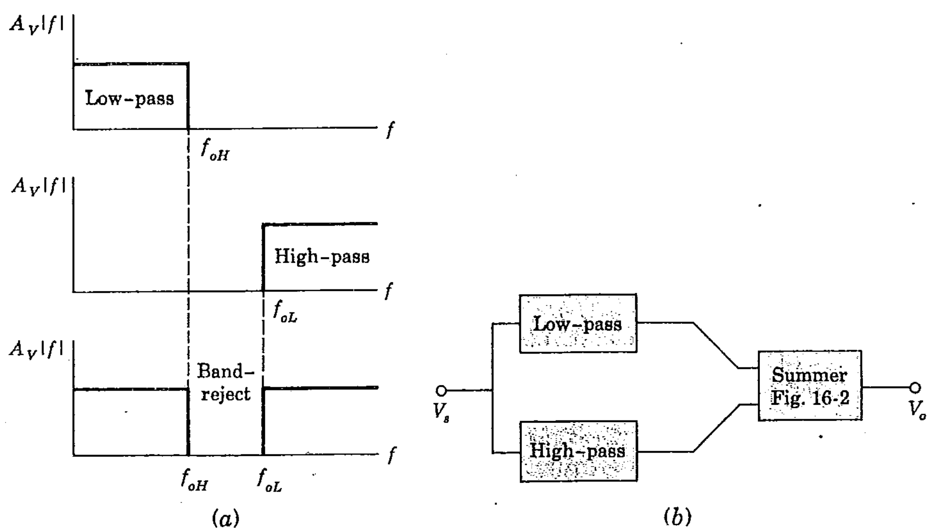

Fig. 16-20 (*a*) Ideal band-reject-filter frequency response. (*b*) Parallel combination of low-pass and high-pass filters results in a band-reject filter.